Chapter 5

Higher level Zhu algebras

We prove that higher level Zhu algebras of a vertex operator algebra are isomorphic to subquotients of its universal enveloping algebra. The main results of this chapter are contained in [He17b].

5.1 The Zhu algebra and higher level Zhu algebras

In this section, we briefly recall the definitions, mainly for fixing the notation. For details, we refer to the papers [DLM98, FHL93, Zhu96].

5.1.1 Vertex operator algebras and their modules

Definition 5.1.1. A *vertex operator algebra* is a vertex algebra $(V, |0\rangle, Y, T)$ with a *conformal vector* or a *Virasoro element* ω, such that if we write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, i.e., $L(n) = \omega_{n+1}$, then

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

for some $c \in \mathbb{C}$, which is called the *central charge* of V. Moreover, $L(-1) = T$ is the infinitesimal translation operator and $L(0)$ is diagonalizable on V, which gives V a $\mathbb{Z}$-grading $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with $L(0)|_{V_n} = n\mathrm{Id}\, V_n$, $\dim\, V_n < \infty$ for all $n \in \mathbb{Z}$ and $V_n = 0$ for $n \ll 0$.

An element $v \in V_n$ is called homogeneous of *conformal weight* n, and we denote it by Δ_v. Whenever we use the notation Δ_v, we assume that v is homogeneous.

Definition 5.1.2. A *weak module* for a vertex operator algebra V is a vector space M, with a linear map $Y_M : V \to \mathrm{End}\, M[[z, z^{-1}]]$ sending v to $Y_M(v, z) = \sum v_n^M z^{-n-1}$ and satisfying:

(1) $Y_M(|0\rangle, z) = \mathrm{Id}_M$ and $Y_M(v, z)w \in M((z))$ for all $v \in V, w \in M,$, i.e., $v_n^M w = 0$ for $n \ll 0$.

(2) For all $\ell, m, n \in \mathbb{Z}$ and $u, v \in V$, we have the Jacobi identity

$$\sum_{i\geq 0}(-1)^i \binom{\ell}{i} \left(u^M_{m+\ell-i}v^M_{n+i} - (-1)^\ell v^M_{n+\ell-i}u^M_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}(u_{\ell+i}v)^M_{m+n-i}.$$

A weak module M is called *admissible* if it has a $\mathbb{Z}_{\geq 0}$-grading $M = \bigoplus_{n \geq 0} M_n$ and satisfies:

(3) For any homogeneous element $v \in V$, we have

$$v_n^M M_m \subseteq M_{m+\Delta_v-n-1}.$$

Submodules, quotient modules, simple modules and semi-simple modules can be defined in the obvious way.

5.1.2 The Zhu algebra

Let $(V, Y, |0\rangle, \omega)$ be a vertex operator algebra. Following [Zhu96], we will construct an associative algebra $\mathrm{Zhu}(V)$ associated to V.

Let

$$O(V) := \mathrm{span}\{u \circ v \mid u, v \in V\},$$

where the linear product $\circ$ is defined on homogeneous $u \in V$ by

$$u \circ v := \mathrm{Res}_z \left(Y(u,z)v \frac{(1+z)^{\Delta_u}}{z^2} \right) = \sum_{i \geq 0} \binom{\Delta_u}{i} u_{i-2} v.$$

Define a product $*$ on V by the formula:

$$u * v := \mathrm{Res}_z \left(Y(u,z)v \frac{(1+z)^{\Delta_u}}{z} \right) = \sum_{i \geq 0} \binom{\Delta_u}{i} u_{i-1} v.$$

The subspace $O(V)$ is known to be a two-sided ideal of V under $*$ [Zhu96].

Let

$$\mathrm{Zhu}(V) := V/O(V).$$

Theorem 5.1.3. *[Zhu96]. The product $*$ induces an associative algebra structure on $\mathrm{Zhu}(V)$ with identity $|0\rangle + O(V)$.*

For an admissible V-module $M = \bigoplus_{n \geq 0} M_n$, we call M_n the n-th level and M_0 the top level of M. Denote by $o^M(u) := u^M_{\Delta_u - 1}$ for all homogeneous $u \in V$ and extend linearly to V. Then $o^M(u) M_n \subseteq M_n$. In particular, $o^M(u)$ preserves the top level. Moreover, the identities

$$o^M(u) o^M(v) = o^M(u * v) \text{ and } o^M(u') = 0$$

hold for all $u, v \in V$ and $u' \in O(V)$ when restricted to the top level M_0. Thus, the top level M_0 is a $\mathrm{Zhu}(V)$-module under the action $(u + O(V)) \cdot m = o^M(u) m$.

The correspondence $M \mapsto M_0$ gives a functor, which we denote by Ω_0, from the category of admissible V-modules to the category of $\mathrm{Zhu}(V)$-modules. On the other hand, Zhu constructed another functor L^0 from the category of $\mathrm{Zhu}(V)$-modules to the category of admissible V-modules in his thesis paper [Zhu96]. Given a $\mathrm{Zhu}(V)$-module U with action π, $L^0(U)$ is an admissible module for V with top level being U. Moreover, we have $\pi(v)m = o^{L^0(U)}(v)m$ for all $m \in U$ and $v \in V$.

Theorem 5.1.4. *[Zhu96]. The two functors Ω_0, L^0 are mutually inverse to each other when restricted to the full subcategory of completely reducible admissible V-modules and the full subcategory of completely reducible $\mathrm{Zhu}(V)$-modules.*

5.1.3 Higher level Zhu algebras

Let $(V, Y, |0\rangle, \omega)$ be a vertex operator algebra. Following [DLM98], we are going to construct an associative algebra $A_n(V)$ for each nonnegative integer n, which we will call the *level n Zhu algebra*[1], with $A_0(V)$ being exactly the Zhu algebra $\mathrm{Zhu}(V)$. We will call the algebras $A_n(V)$ *higher level Zhu algebras* when $n \geq 1$.

Recall that $L(n) = \omega_{n+1}$, where ω is the Virasoro element of V. For $n \geq 0$, let

$$O_n(V) := \mathrm{span}\{u \circ_n v, L(-1)u + L(0)u \mid u, v \in V\},$$

where the linear product $\circ_n$ is defined on homogeneous $u \in V$ by

$$\begin{aligned} u \circ_n v : &= \mathrm{Res}_z \left(Y(u,z)v \frac{(1+z)^{\Delta_u + n}}{z^{2n+2}} \right) \\ &= \sum_{i=0}^{\infty} \binom{\Delta_u + n}{i} u_{i-2n-2} v. \end{aligned}$$

Define a product $*_n$ on V by the formula:

$$\begin{aligned} u *_n v := & \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_z \left(Y(u,z)v \frac{(1+z)^{\Delta_u + n}}{z^{n+m+1}} \right) \\ = & \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\Delta_u + n}{i} u_{i-m-n-1} v. \end{aligned}$$

The subspace $O_n(V)$ is a two-sided ideal of V under $*_n$ [DLM98].

Let

$$A_n(V) := V/O_n(V).$$

Theorem 5.1.5. *[DLM98]. The product $*_n$ induces an associative algebra structure on $A_n(V)$ with identity $|0\rangle + O_n(V)$. Moreover, the identity map on V induces a surjective algebra homomorphism from $A_n(V)$ to $A_{n-1}(V)$ for $n \geq 1$.*

[1] We follow the terminology as in [vE11] for the twisted case.

Remark 5.1.6. *Note that* $L(-1)u + L(0)u = u \circ |0\rangle$ *and* $u \circ_0 v = u \circ v$*, so* $O_0(V)$ *coincides with* $O(V)$*. Moreover, as* $u *_0 v = u * v$*, the algebra* $A_0(V) = \mathrm{Zhu}(V)$ *is just the Zhu algebra.*

We have an inverse system of associative algebras:

$$A_0(V) \twoheadleftarrow A_1(V) \twoheadleftarrow \cdots \twoheadleftarrow A_n(V) \twoheadleftarrow A_{n+1}(V) \twoheadleftarrow \cdots . \tag{5.1}$$

These higher level Zhu algebras play similar roles to that of the Zhu algebra in the representation theory of vertex operator algebras. To describe the relationship between the representations of $A_n(V)$ and those of V, we recall a Lie algebra associated to V.

Consider the vector space $V \otimes \mathbb{C}[t, t^{-1}]$ and the linear operator

$$\partial := L(-1) \otimes \mathrm{Id} + \mathrm{Id} \otimes \frac{d}{dt}.$$

Let

$$\hat{V} := \frac{V \otimes \mathbb{C}[t, t^{-1}]}{\partial(V \otimes \mathbb{C}[t, t^{-1}])}.$$

Denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$ for $v \in V$ and $m \in \mathbb{Z}$. The vector space $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining the degree of $v(m)$ to be $\Delta_v - m - 1$ and the Lie bracket:

$$[u(m), v(n)] = \sum_{i \geq 0} \binom{m}{i} (u_i v)(m + n - i) \text{ for } u, v \in V. \tag{5.2}$$

As the Lie bracket (5.2) in $\hat{V}$ is just the commutator formula (4.2) in V, the natural map from $\hat{V}$ to $\mathrm{End}\, V$ sending $v(m)$ to v_m is a Lie algebra homomorphism. In this way, we can consider a V-module as a $\hat{V}$-module.

Denote the homogeneous subspace of $\hat{V}$ of degree m by $\hat{V}(m)$. Then $\hat{V}(0)$ is a Lie subalgebra of $\hat{V}$. Consider the Lie algebra structure of $A_n(V)$ with Lie bracket $[u, v] = u *_n v - v *_n u$ for $u, v \in V$. One can show that [DLM98] there is a surjective Lie algebra homomorphism from $\hat{V}(0)$ to $A_n(V)$ for each n, sending $o(v) := v(\Delta_v - 1)$ to $v + O_n(V)$. Let $U(\hat{V})$ be the universal enveloping algebra of $\hat{V}$. Then it inherits a natural $\mathbb{Z}$-grading from $\hat{V}$, say $U(\hat{V}) = \bigoplus_{n \in \mathbb{Z}} U(\hat{V})_n$.

Let $P_n = \bigoplus_{i > n} \hat{V}(i) \oplus \hat{V}(0)$. Given an $A_n(V)$-module N, we can consider it as a $\hat{V}(0)$-module, and then as a P_n-module by letting $\bigoplus_{i > n} \hat{V}(i)$ act trivially. Define

$$M_n(N) = \mathrm{Ind}_{P_n}^{\hat{V}}(N) = U(\hat{V}) \otimes_{U(P_n)} N.$$

By setting the degree of N to be n, the $\mathbb{Z}$-gradation of $\hat{V}$ lifts to $M_n(N)$ with $M_n(N)(i) = U(\hat{V})_{i-n} N$. Let W be the subspace of $M_n(N)$ spanned by the coefficients of (where $u, v \in V, m \in M_n(N)$)

$$(z + w)^{\Delta_u + n} Y(u, z + w) Y(v, w) m - (w + z)^{\Delta_u + n} Y(Y(u, z)v, w) m.$$

Let

$$\overline{M}_n(N) := M_n(N) / U(\hat{V}) W. \tag{5.3}$$

Theorem 5.1.7 ([DLM98]). *The space* $\overline{M}_n(N) = \sum_{i\geq 0} \overline{M}_n(N)(i)$ *admits an admissible* V*-module structure with* $\overline{M}_n(N)(0) \neq 0$ *and* $\overline{M}_n(N)(n) = N$.

Let $M = \bigoplus_{i\geq 0} M_i$ be an admissible V-module and n a nonnegative integer, and define the subspace

$$\Omega_n(M) := \{m \in M \mid \hat{V}(-k)m = 0 \text{ if } k > n\}.$$

Then one can show that $\Omega_n(M)$ admits an $A_n(V)$-module structure under the action $v \cdot m = o^M(v)m$, with each M_i being a submodule for $0 \leq i \leq n$. The module $\overline{M}_n(N)$ has the universal property that if W is any weak V-module, and $\varphi : N \to \Omega_n(W)$ any $A_n(V)$-module homomorphism, then there is a unique V-module homomorphism $\tilde{\varphi} : \overline{M}_n(N) \to W$ which extends φ [DLM98].

Since there is a surjective homomorphism $A_n(V) \twoheadrightarrow A_{n-1}(V)$, the subspace $\Omega_{n-1}(M) \subseteq \Omega_n(M)$ is naturally an $A_n(V)$-module. Let

$$\Omega_n/\Omega_{n-1}(M) := \frac{\Omega_n(M)}{\Omega_{n-1}(M)}.$$

Then Ω_n/Ω_{n-1} defines a functor from the category of admissible V-modules to the category of $A_n(V)$-modules. The good thing is that this functor has an inverse when restricted to an appropriate subcategory. In [DLM98], the authors constructed a functor L^n from the category of $A_n(V)$-modules to the category of admissible V-modules, such that, for a given $A_n(V)$-module N with action π, if N itself and its proper submodules do not factor through $A_{n-1}(V)$ (this condition was added in [BVY17]), then $\Omega_n/\Omega_{n-1}(L^n(N)) \cong N$ as $A_n(V)$-modules, i.e., $o^{L^n(U)}(v)m = \pi(v)m$ for all $v \in V$ and $m \in N$.

Theorem 5.1.8. *[DLM98, BVY17]. The functors* Ω_n/Ω_{n-1} *and* L^n *are inverse to each other when restricted to the full subcategory of completely reducible admissible* V*-modules that are generated by their degree* n *subspace and the full subcategory of completely reducible* $A_n(V)$*-modules whose irreducible components do not factor through* $A_{n-1}(V)$.

5.2 The universal enveloping algebra and its subquotients

To define the universal enveloping algebra of a vertex operator algebra, we need to introduce a completion notation, as the Jacobi identity contains infinite sums.

Recall that the Lie algebra $\hat{V}$ that we constructed in the previous section is $\mathbb{Z}$-graded. The zero component $U(\hat{V})_0$ of $U(\hat{V})$ contains $U(\hat{V}(0))$, the universal enveloping algebra of $\hat{V}(0)$, as a subalgebra. For $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\leq 0}$, let

$$U(\hat{V})_n^k = \sum_{i\leq k} U(\hat{V})_{n-i}U(\hat{V})_i \quad \text{and} \quad U(\hat{V}(0))^k = U(\hat{V}(0)) \cap U(\hat{V})_0^k.$$

Then

$$\cdots \subseteq U(\hat{V})_n^k \subseteq U(\hat{V})_n^{k+1} \subseteq \cdots \subseteq U(\hat{V})_n^0 = U(\hat{V})_n$$

and

$$\cdots \subseteq U(\hat{V}(0))^k \subseteq U(\hat{V}(0))^{k+1} \subseteq \cdots \subseteq U(\hat{V}(0))^0 = U(\hat{V}(0))$$

are well-defined filtrations of $U(\hat{V})_n$ and $U(\hat{V}(0))$, respectively. Moreover, we have

$$\bigcap_k U(\hat{V})_n^k = 0, \qquad \bigcup_k U(\hat{V})_n^k = U(\hat{V})_n.$$

Hence, the filtration $\{U(\hat{V})_n^k\}_{k\leq 0}$ forms a fundamental neighborhood system of $U(\hat{V})_n$. Let $\tilde{U}(\hat{V})_n$ be the completion of $U(\hat{V})_n$ with respect to this filtration, i.e., infinite sums are allowed in $\tilde{U}(\hat{V})_n$, and for any given k, only finitely many terms are contained in $U(\hat{V})_n^{k+1} \setminus U(\hat{V})_n^k$. Let $\tilde{U}(\hat{V}(0))$ be the completion of $U(\hat{V}(0))$ with respect to the filtration $\{U(\hat{V}(0))^k\}_{k\leq 0}$. It is obviously a subspace of $\tilde{U}(\hat{V})_0$.

Let

$$\tilde{U}(\hat{V}) := \bigoplus_{n\in\mathbb{Z}} \tilde{U}(\hat{V})_n.$$

The space $\tilde{U}(\hat{V})$ becomes a $\mathbb{Z}$-graded ring with each component $\tilde{U}(\hat{V})_n$ being complete. The subspace $U(\hat{V})$ is a dense subalgebra of $\tilde{U}(\hat{V})$ with $U(\hat{V})_n$ being dense in $\tilde{U}(\hat{V})_n$ for all n. The completion $\tilde{U}(\hat{V})$ is called a degreewise completed topological ring in the theory of quasi-finite algebras studied by A. Matsuo et al. in [MNT10].

Consider the relations

$$\begin{aligned}
\langle \text{Vac} \rangle: \quad & |0\rangle(i) = \delta_{i,-1}, \text{ for all } i \in \mathbb{Z}, \\
\langle \text{Vir} \rangle: \quad & [L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}\mathbf{c}, \text{ for all } m, n \in \mathbb{Z}, \\
J^{u,v}_{m,n,\ell}: \quad & \sum_{i\geq 0}(-1)^i \binom{\ell}{i} \Big(u(m+\ell-i)v(n+i) - (-1)^\ell v(n+\ell-i)u(m+i)\Big) \\
& = \sum_{i\geq 0} \binom{m}{i} (u_{\ell+i}v)(m+n-i), \text{ for } u, v \in V \text{ and } m, n, \ell \in \mathbb{Z}.
\end{aligned}$$

Remark 5.2.1. *The element $L(n)$ should be considered as the image of $\omega \otimes t^{n+1}$ in $\hat{V}$. The Jacobi relation $J^{u,v}_{m,n,\ell}$ is now well-defined in $\tilde{U}(\hat{V})$.*

Definition 5.2.2. The *universal enveloping algebra* $U(V)$ of V is the quotient of $\tilde{U}(\hat{V})$ by the relations: $\langle\text{Vac}\rangle$, $\langle\text{Vir}\rangle$ and $\langle J^{u,v}_{m,n,\ell} \mid u, v \in V, m, n, \ell \in \mathbb{Z}\rangle$.

Remark 5.2.3. *The universal enveloping algebra $U(V)$ of a vertex operator algebra V is an associative algebra, while the universal enveloping vertex algebra $V(R)$ of a non-linear Lie conformal algebra R that we defined in Definition 4.2.8 is a vertex algebra.*

All the relations $\langle\text{Vac}\rangle$, $\langle\text{Vir}\rangle$ and $J^{u,v}_{m,n,\ell}$ are homogeneous, so the universal enveloping algebra $U(V)$ inherits a natural $\mathbb{Z}$-grading from $\tilde{U}(\hat{V})$.

The image of $\tilde{U}(\hat{V}(0))$ in $U(V)$ is obviously contained in $U(V)_0$, which we denote by $U(V(0))$ and is a subalgebra of $U(V)$.

Let

$$U(V)_0^k := \sum_{i\le k} U(V)_{-i} U(V)_i \quad \text{and} \quad U(V(0))^k := U(V(0)) \cap U(V)_0^k.$$

Then $\dfrac{U(V)_0}{U(V)_0^k}$ and $\dfrac{U(V(0))}{U(V(0))^k}$ inherit associative algebra structures, as $U(V)_0^k$ and $U(V(0))^k$ are two-sided ideals of $U(V)_0$ and $U(V(0))$, respectively. By the obvious inclusions $U(V)_0^k \subseteq U(V)_0^{k+1}$ and $U(V(0))^k \subseteq U(V(0))^{k+1}$, we have two inverse systems of algebras:

$$\frac{U(V)_0}{U(V)_0^{-1}} \twoheadleftarrow \frac{U(V)_0}{U(V)_0^{-2}} \twoheadleftarrow \cdots \twoheadleftarrow \frac{U(V)_0}{U(V)_0^{-n}} \twoheadleftarrow \frac{U(V)_0}{U(V)_0^{-n-1}} \twoheadleftarrow \cdots,$$
$$\frac{U(V(0))}{U(V(0))^{-1}} \twoheadleftarrow \frac{U(V(0))}{U(V(0))^{-2}} \twoheadleftarrow \cdots \twoheadleftarrow \frac{U(V(0))}{U(V(0))^{-n}} \twoheadleftarrow \frac{U(V(0))}{U(V(0))^{-n-1}} \twoheadleftarrow \cdots.$$

Our goal is prove that these two inverse systems of associative algebras are both isomorphic to the inverse system given by higher level Zhu algebras (5.1). More precisely, we are going to prove that

$$A_n(V) \cong \frac{U(V)_0}{U(V)_0^{-n-1}} \cong \frac{U(V(0))}{U(V(0))^{-n-1}} \text{ for } n \ge 0.$$

5.3 The isomorphisms

One of our motivations for this study is the paper [FZ92] of I. Frenkel and Y. C. Zhu, where they observed that the Zhu algebra is isomorphic to a subquotient of the universal enveloping algebra. In this section, we prove that all higher level Zhu algebras are also isomorphic to subquotients of the universal enveloping algebra.

For simplicity, we use the following notation: For $u, v \in V$ and $m, n, \ell \in \mathbb{Z}$, let

$$ {}^1J^{u,v}_{m,n,\ell} := \sum_{i\ge 0} \binom{m}{i} (u_{\ell+i} v)(m+n-i),$$
$$ {}^2J^{u,v}_{m,n,\ell} := \sum_{i\ge 0} (-1)^i \binom{\ell}{i} \left(u(m+\ell-i)v(n+i) - (-1)^\ell v(n+\ell-i)v(m+i)\right).$$

They are just the two sides of the Jacobi identity $J^{u,v}_{m,n,\ell}$, so in the universal enveloping algebra $U(V)$, we have ${}^1J^{u,v}_{m,n,\ell} = {}^2J^{u,v}_{m,n,\ell}$.

We use the following notation, which is defined for homogeneous elements and extended linearly to all of V.

$$J_n(u) := u(\Delta_u - 1 + n).$$

A good property of this notation is that the degree of $J_n(u)$ is always $-n$.

Let

$$
\begin{aligned}
{}^{(1)}J^{u,v}_{m,n,\ell} :=&\ {}^{1}J^{u,v}_{m+\Delta_u-1,n+\Delta_v-1,\ell} \\
=&\sum_{i\geq 0}\binom{m+\Delta_u-1}{i}J_{m+n+\ell}(u_{\ell+i}v), \\
{}^{(2)}J^{u,v}_{m,n,\ell} :=&\ {}^{2}J^{u,v}_{m+\Delta_u-1,n+\Delta_v-1,\ell} \\
=&\sum_{i\geq 0}(-1)^i\binom{\ell}{i}(J_{m+\ell-i}(u)J_{n+i}(v)-(-1)^{\ell}J_{n+\ell-i}(v)J_{m+i}(u)).
\end{aligned}
$$

Every term in the expressions ${}^{(1)}J^{u,v}_{m,n,\ell}$, ${}^{(2)}J^{u,v}_{m,n,\ell}$ is of the same degree $-m-n-\ell$.

For a negative integer n and a positive integer k, recall that

$$
\binom{n}{k}=\frac{n(n-1)\cdots(n-k+1)}{k!}=(-1)^k\binom{-n+k-1}{k}. \tag{5.4}
$$

The statement of the following lemma was suggested by Atsushi Matsuo.

Lemma 5.3.1. *For any integers s,t and N satisfying $N+s\geq 0$,*

$$
\begin{aligned}
X :=& \sum_{j=0}^{N}\binom{-N-s-1}{j}\,{}^{(2)}J^{u,v}_{N+1,t+j,-N-s-1-j} \\
=&\ J_{-s}(u)J_t(v)+\sum_{k\geq N+1}\sum_{j=0}^{N}(-1)^j\binom{N+s+j}{j}\binom{N+s-k}{k-j}J_{-k-s}(u)J_{k+t}(v) \\
&-\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^{N+s+1}\binom{N+s+j}{j}\binom{N+s+j+i}{i}J_{t-N-s-1-i}(v)J_{N+1+i}(u).
\end{aligned}
$$

Proof. By definition, ${}^{(2)}J^{u,v}_{N+1,t+j,-N-s-1-j}=A-B$, where

$$
\begin{aligned}
A =& \sum_{i\geq 0}(-1)^i\binom{-N-s-1-j}{i}J_{-s-j-i}(u)J_{t+j+i}(v), \\
B =& \sum_{i\geq 0}(-1)^{-N-s-1-j+i}\binom{-N-s-1-j}{i}J_{t-N-s-1-i}(v)J_{N+1+i}(u).
\end{aligned}
$$

Therefore, $X = C - D$, where

$$\begin{aligned}
C &= \sum_{j=0}^{N} \binom{-N-s-1}{j} A \\
&= \sum_{j=0}^{N} \sum_{i\geq 0} (-1)^j \binom{N+s+j}{j} \binom{N+s+j+i}{i} J_{-s-j-i}(u) J_{t+j+i}(v), \\
D &= \sum_{j=0}^{N} \binom{-N-s-1}{j} B \\
&= \sum_{j=0}^{N} \sum_{i\geq 0} (-1)^{N+s+1} \binom{N+s+j}{j} \binom{N+s+i+j}{i} J_{t-N-s-1-i}(v) J_{N+1+i}(u).
\end{aligned}$$

We used the formula (5.4) in the above calculation.

Let $k = i + j$ in the expression of C. Then

$$\begin{aligned}
C &= \sum_{j=0}^{N} \sum_{k\geq j} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v) \\
&= \sum_{k=0}^{N} \sum_{j=0}^{k} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v) \\
&\quad + \sum_{k\geq N+1} \sum_{j=0}^{N} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v).
\end{aligned} \tag{5.5}$$

In the expression (5.5), for $1 \leq k \leq N$, we have

$$\begin{aligned}
&\sum_{j=0}^{k} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = \sum_{j=0}^{k} (-1)^j \frac{(N+s+j)!}{j!(N+s)!} \frac{(N+s+k)!}{(k-j)!(N+s+j)!} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = \sum_{j=0}^{k} (-1)^j \frac{(N+s+k)!}{(N+s)!k!} \frac{k!}{(k-j)!j!} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = \binom{N+s+k}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = 0,
\end{aligned}$$

and for $k = 0$, we will have $j = 0$, so only one term will be left in (5.5), namely, $J_{-s}(u)J_t(b)$. □

Corollary 5.3.2. In the universal enveloping algebra $U(V)$, for any integers s, t and N satisfying

$N+s\geq 0$, we have the identity

$$
\begin{aligned}
&J_{-s}(u)J_t(v)\\
&=\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^i\binom{N+\Delta_u}{i}\binom{-N-s-1}{j}J_{t-s}(u_{-N-s-i-j-1}v)\\
&\quad-\sum_{k\geq N+1}\sum_{j=0}^{N}(-1)^j\binom{N+s+j}{j}\binom{N+s-k}{k-j}J_{-k-s}(u)J_{k+t}(v)\\
&\quad+\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^{N+s+1}\binom{N+s+j}{j}\binom{N+s+j+i}{i}J_{t-N-s-1-i}(v)J_{N+1+i}(u).
\end{aligned}
$$

Proof. In the universal enveloping algebra, we have

$$
\begin{aligned}
&\sum_{j=0}^{N}\binom{-N-s-1}{j}{}^{(2)}J^{u,v}_{N+1,t+j,-N-s-1-j}\\
&=\sum_{j=0}^{N}\binom{-N-s-1}{j}{}^{2}J^{u,v}_{N+1+\Delta_u,t+j+\Delta_v,-N-s-1-j}\\
&=\sum_{j=0}^{N}\binom{-N-s-1}{j}{}^{1}J^{u,v}_{N+1+\Delta_u,t+j+\Delta_v,-N-s-1-j}\\
&=\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^i\binom{-N-s-1}{j}\binom{N+\Delta_u}{i}J_{t-s}(u_{-N-s-i-j-1}v).
\end{aligned}
$$

The desired identity then follows from Lemma 5.3.1. □

The following lemma will be very important in the proof of Theorem 5.3.4.

Lemma 5.3.3. *Let $n\geq 0$. Then every element $\sum J_{n_1}(u_1)\cdots J_{n_m}(u_m)$ in $\frac{U(V)_0}{U(V)_0^{-n}}$ can be expressed as $J_0(u(w))$ for some $u(w)\in V$.*

Proof. We only need to prove the claim for monomials $w=J_{n_1}(u_1)\cdots J_{n_m}(u_m)$. Define the degree of w to be m, i.e., the number of factors of it. Then a degree one element in $U(V)_0$ is just an element of the form $J_0(u)$ for some $u\in V$, and we need to show that every monomial in the quotient $\frac{U(V)_0}{U(V)_0^{-n}}$ is congruent to a degree one element.

We use induction on the degree of the monomial w. If $m=1$, there is nothing to do. Let $m=k\geq 2$ and assume that for every monomial of degree less than k, it is congruent to a degree one element in the quotient $\frac{U(V)_0}{U(V)_0^{-n}}$.

Use the formula in Corollary 5.3.2 for $J_{n_{m-1}}(u_{m-1})J_{n_m}(u_m)$, where

$$-s=n_{m-1},\ t=n_m,\ u=u_{m-1},\ v=u_m.$$

In the statement of Corollary 5.3.2, choose N sufficiently large, so that $\min\{N+n_m, N\} > n$. Then $J_{k+n_m}(u_m)$ and $J_{N+1+i}(u_{m-1})$ are both contained in $\bigoplus_{j\leq -n} U(V)_j$ for $k \geq N+1$, and $w = J_{n_1}(u_1)\cdots J_{n_m}(u_m)$ is congruent to a linear combination of the following lower degree monomials:

$$J_{n_1}(u_1)\cdots J_{n_{m-2}}(u_{m-2})J_{n_m+n_{m-1}}((u_{m-1})_{-N+n_{m-1}-i-j-1}u_m).$$

By induction, these lower degree monomials are congruent to degree one monomials, so w is itself congruent to a degree one monomial. □

Now we are in a position to prove the isomorphisms between higher level Zhu algebras and subquotients of the universal enveloping algebra.

Theorem 5.3.4. *For $n \geq 0$, we have the isomorphism*

$$A_n(V) \cong \frac{U(V)_0}{U(V)_0^{-n-1}}. \tag{5.6}$$

Proof. Let φ be the map from V to $U(V)_0$ sending v to $o(v)$, where $o(v)$ is the image of $v(\Delta_v - 1)$ in $U(V)$ for homogeneous v and extended linearly to V. Combine it with the canonical quotient map from $U(V)_0$ to $\frac{U(V)_0}{U(V)_0^{-n-1}}$. Then Lemma 5.3.3 tells us that this map is surjective.

First, we show that φ factors through $A_n(V)$, i.e., $\varphi(O_n(V)) \subseteq U(V)_0^{-n-1}$.

Recall that $O_n(V) = \operatorname{span}\{u \circ_n v, L(-1)u + L(0)u \mid u, v \in V, u \text{ homogeneous}\}$, where

$$u \circ_n v = \sum_{i=0}^{\Delta_u+n} \binom{\Delta_u+n}{i} u_{i-2n-2}v.$$

As $\varphi(L(-1)u + L(0)u) \equiv 0$, we only need to prove that $\varphi(u \circ_n v) \in U(V)_0^{-n-1}$. Assume that u, v are both homogeneous. Then $\Delta_{u_{i-2n-2}v} = \Delta_u + \Delta_v + 2n + 1 - i$, and

$$\begin{aligned}
\varphi(u \circ_n v) &= \sum_{i=0}^{\Delta_u+n} \binom{\Delta_u+n}{i} (u_{i-2n-2}v)(\Delta_u+\Delta_v+2n-i) \\
&= {}^{(1)} J^{u,v}_{n+1,n+1,-2n-2} \\
&= {}^{(2)} J^{u,v}_{n+1,n+1,-2n-2} \\
&= \sum_{i\geq 0} (-1)^i \binom{-2n-2}{i} J_{-n-1-i}(u)J_{n+1+i}(v) \\
&\quad - \sum_{i\geq 0} (-1)^i \binom{-2n-2}{i} J_{-n-1-i}(v)J_{n+1+i}(u).
\end{aligned}$$

As $\deg\ J_{n+1+i}(v) = \deg\ J_{n+1+i}(u) \leq -n-1$, we have $\varphi(u \circ_n v) \in U(V)_0^{-n-1}$.

Next we prove that φ is an algebra homomorphism, i.e., $\varphi(u *_n v) = \varphi(u)\varphi(v)$.

Recall that

$$u *_n v = \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\Delta_u + n}{i} u_{i-m-n-1} v.$$

We have

$$\begin{aligned} &\varphi(u *_n v) \\ &= \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\Delta_u + n}{i} (u_{i-m-n-1} v)(\Delta_u + \Delta_v + m + n - i). \end{aligned}$$

By letting $s = t = 0$ and $N = n$ in Corollary 5.3.2, we have

$$\begin{aligned} J_0(u) J_0(v) &\equiv \sum_{j=0}^{n} \binom{-n-1}{j} J^{(1)}_{n+1,j,-n-1-j}(u,v) \quad \text{mod } U(V)_0^{-n-1} \\ &\equiv \sum_{j=0}^{n} \sum_{i \geq 0} \binom{-n-1}{j} \binom{\Delta_u + n}{i} J_0(u_{-n-1+i-j}) \quad \text{mod } U(V)_0^{-n-1} \\ &\equiv \sum_{j=0}^{n} \sum_{i=0}^{\infty} (-1)^j \binom{n+j}{j} \binom{\Delta_u + n}{i} J_0(u_{i-j-n-1} v) \quad \text{mod } U(V)_0^{-n-1}, \end{aligned}$$

that is, $\varphi(u *_n v) = \varphi(u)\varphi(v)$.

Finally, we want to construct an inverse map for φ. By Lemma 5.3.3, every element of $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$ can be expressed as $J_0(u) + U(V)_0^{-n-1}$ for some $u \in V$. We want to define the map φ^{-1} from $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$ to $A_n(V)$ sending $J_0(u) + U(V)_0^{-n-1}$ to $u + O_n(V)$. Once we prove that this is a well-defined map, it is an inverse for φ. The well-definedness requires that whenever $J_0(u) \in U(V)_0^{-n-1}$, we have $u \in O_n(V)$, i.e., φ^{-1} does not depend on the representatives of an element of $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$. Consider the induced module $\overline{M}(A_n(V))$ constructed in (5.3), where $A_n(V)$ is the regular module of $A_n(V)$. If $J_0(u) \in U(V)_0^{-n-1}$, then $J_0(u)$ will kill the subspace $\overline{M}_n(A_n(V))(n)$, which by Theorem 5.1.7 is isomorphic to $A_n(V)$ itself as $A_n(V)$-modules. Therefore, $J_0(u)v = u *_n v$ for all $v \in V$. In particular, for $v = |0\rangle$, which is the identity element of $A_n(V)$, we have $u *_n |0\rangle = J_0(u)|0\rangle = 0$, which implies that $u \in O_n(V)$. □

Corollary 5.3.5. The Zhu algebra is isomorphic to a subquotient of the universal enveloping algebra,

$$\text{Zhu}(V) = A_0(V) \cong \frac{U(V)_0}{U(V)_0^{-1}}.$$

Recall that there is a surjective Lie algebra homomorphism from $\hat{V}(0)$ to $A_n(V)$, which induces a surjective associative algebra homomorphism from $U(\hat{V}(0))$ to $A_n(V)$. Composing it with the isomorphism (5.6), we can conclude that $U(\hat{V}(0))$ is a dense subalgebra of $U(V)_0$.

Corollary 5.3.6. The subalgebra $U(\hat{V}(0))$ is dense in $U(V)_0$, i.e., $U(\hat{V}(0)) + U(V)_0^{-n} = U(V)_0$ for all $n \geq 0$, hence we have isomorphisms:

$$\frac{U(V)_0}{U(V)_0^{-n-1}} \cong \frac{U(V(0))}{U(V(0))^{-n-1}}.$$

Let $C_2V := \mathrm{span}\{u_{-2}v \mid u, v \in V\}$. A vertex operator algebra V is called C_2*-cofinite* if $\dim \frac{V}{C_2V} < \infty$. In [MNT10], the authors proved that if V is C_2-cofinite, then all the subquotients $\frac{U(V)_0}{U(V)_0^{-n-1}}$ are finite dimensional. With the isomorphisms between $A_n(V)$ and these subquotients, we easily get the corollary below.

Corollary 5.3.7. If V is a C_2-cofinite vertex operator algebra, then all of its higher level Zhu algebras are finite dimensional.