Chapter 3

Semi-infinite cohomology

In this chapter, we develop an adjusted version of semi-infinite cohomology which will be used to define affine W-algebras in Chapter 4. The main results of this chapter are contained in [He17a].

3.1 A brief review of Lie algebra cohomology

Let L be a complex Lie algebra and M be an L-module. The space of n-cochains (or n-forms) with coefficients in M is the space $C^n(L, M) := \mathrm{Hom}_{\mathbb{C}}(\Lambda^n L, M)$, where $\Lambda^n L$ is the n-th exterior power of L. Given an n-cochain $f \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^n L, M)$, the coboundary of f is the $(n+1)$-cochain δf, defined to be

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{n+1}) = &\sum_{i=1}^{n+1} (-1)^i x_i \cdot f(x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}),\end{aligned} \tag{3.1}$$

where $\hat{x}_i$ means that the term x_i is omitted and $\cdot$ is the Lie algebra action on M. One can show by straightforward calculations that $\delta^2 = 0$, hence we have a complex $(C^\bullet(L, M), \delta)$.

Definition 3.1.1. The complex $(C^\bullet(L, M), \delta)$ is called the *Chevalley-Eilenberg cochain complex* and its cohomology is called the *cohomology of L with coefficients in M*.

Let $L^* = \mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C})$ be the dual of L. Assume that L is finite-dimensional, while $\{e_1, \cdots, e_d\}$ and $\{e_1^*, \cdots, e_d^*\}$ are well-ordered dual bases of L and L^*, respectively, in the sense that $\langle e_i^*, e_j \rangle = \delta_{i,j}$. One can identify $\mathrm{Hom}_{\mathbb{C}}(\Lambda^n L, M)$ with $\Lambda^n L^* \otimes M$ by considering $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes m$ as the n-cochain sending $e_{j_1} \wedge \cdots \wedge e_{j_n}$ to $\det(\langle e_{i_k}^*, e_{j_\ell} \rangle)_{1 \le k, \ell \le n} m$. If we assume that in the above expressions we have $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$, then

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes m)(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \begin{cases} m & \text{if } i_1 = j_1, \cdots, i_n = j_n, \\ 0 & \text{otherwise.} \end{cases}$$

The Clifford algebra $Cl(L \oplus L^*)$ is the associative algebra generated by $\{\iota(e_i), \varepsilon(e_i^*)\}_{1 \le i \le d}$, with relations:

$$\iota(e_i)\iota(e_j) + \iota(e_j)\iota(e_i) = \varepsilon(e_j^*)\varepsilon(e_i^*) + \varepsilon(e_i^*)\varepsilon(e_j^*) = 0 \text{ and } \iota(e_i)\varepsilon(e_j^*) + \varepsilon(e_j^*)\iota(e_i) = \delta_{i,j}. \quad (3.2)$$

The Clifford algebra $Cl(L \oplus L^*)$ acts on $\Lambda^\bullet L^* = \bigoplus_{i \ge 0} \Lambda^i L^*$ in the following way: $\iota(e_i)$ is the contraction operator $\iota(e_i) : \Lambda^n L^* \to \Lambda^{n-1} L^*$ defined by

$$\iota(e_i) \cdot y_1^* \wedge \cdots \wedge y_n^* = \sum_k (-1)^{k+1} \langle y_k^*, e_i \rangle y_1^* \wedge \cdots \wedge \hat{y}_k^* \wedge \cdots \wedge y_n^*,$$

and $\varepsilon(e_i^*)$ is the wedging operator $\varepsilon(e_i^*) : \Lambda^n L^* \to \Lambda^{n+1} L^*$ defined by

$$\varepsilon(e_i^*) \cdot y_1^* \wedge \cdots \wedge y_n^* = e_i^* \wedge y_1^* \wedge \cdots \wedge y_n^*.$$

Straightforward calculations show that these operators $\iota(e_i)$ and $\varepsilon(e_i^*)$ satisfy (3.2), so it defines an action of $Cl(L \oplus L^*)$ on $\Lambda^\bullet L^*$.

Let

$$\bar{\delta} = \sum_i \varepsilon(e_i^*) \otimes e_i - \sum_{i<j} \varepsilon(e_i^*)\varepsilon(e_j^*)\iota([e_i, e_j]) \otimes 1. \quad (3.3)$$

Then $\bar{\delta} \in Cl(L \oplus L^*) \otimes U(L)$, hence it has a well-defined action on $\Lambda^\bullet L^* \otimes M$.

Proposition 3.1.2. *The operator $\bar{\delta}$ defined by (3.3) realizes the operator δ defined by (3.1) in the Chevalley-Eilenberg complex.*

Proof. We need to show that $\bar{\delta} f = \delta f$ for all $f \in \Lambda^\bullet L^* \otimes M$. It is clear that both $\bar{\delta}$ and δ map $\Lambda^n L^* \otimes M$ to $\Lambda^{n+1} L^* \otimes M$. Thus we only need to prove that for $f = e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes m \in \Lambda^n L^* \otimes M$ and $\omega = e_{j_1} \wedge \cdots \wedge e_{j_{n+1}} \in \Lambda^{n+1} L$, we have $(\bar{\delta} f)(\omega) = (\delta f)(\omega)$. We assume that $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_{n+1}$. By definition,

$$\begin{aligned}(\delta f)(\omega) = &\sum_{\ell=1}^{n+1} (-1)^\ell e_{j_\ell} \cdot f(e_{j_1}, \cdots, \hat{e}_{j_\ell}, \cdots, e_{j_{n+1}}) \\ &+ \sum_{1 \le k < \ell \le n+1} (-1)^{k+\ell} f([e_{j_k}, e_{j_\ell}], e_{j_1}, \cdots, \hat{e}_{j_k}, \cdots, \hat{e}_{j_\ell}, \cdots, e_{j_{n+1}}).\end{aligned}$$

Note that

$$\sum_k \varepsilon(e_k^*) \otimes e_k \cdot f = \sum_k e_k^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes e_k \cdot m,$$

and

$$\begin{aligned}&(e_k^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes e_k \cdot m)(\omega) \\ &\quad = \begin{cases} (-1)^\ell e_{j_\ell} \cdot f(e_{j_1}, \cdots, \hat{e}_{j_\ell}, \cdots, e_{j_{n+1}}) & \text{if } k = j_\ell, \\ 0 & \text{if } k \notin \{j_1, \cdots, j_{n+1}\}, \end{cases}\end{aligned}$$

so

$$\left(\sum_k \varepsilon(e_k^*)\otimes e_k\cdot f\right)(\omega)=\sum_{\ell=1}^{n+1}(-1)^\ell e_{j_\ell}\cdot f(e_{j_1},\cdots,\hat{e}_{j_\ell},\cdots,e_{j_{n+1}}).$$

Let $f_{\hat{i}_s}=e_{i_1}^*\wedge\cdots\wedge\hat{e}_{i_s}^*\cdots\wedge e_{i_n}^*\otimes m$ and $\omega_{\hat{j}_k,\hat{j}_\ell}=e_{j_1}\wedge\cdots\wedge\hat{e}_{j_k}\cdots\wedge\hat{e}_{j_\ell}\cdots\wedge e_{j_{n+1}}$. Then

$$\varepsilon(e_i^*)\varepsilon(e_j^*)\iota([e_i,e_j])\otimes 1\cdot f=\sum_{1\le s\le n}(-1)^{s+1}\langle e_{i_s}^*,[e_i,e_j]\rangle e_i^*\wedge e_j^*\wedge f_{\hat{i}_s},$$

and

$$(e_i^*\wedge e_j^*\wedge f_{\hat{i}_s})(\omega)=\begin{cases}(-1)^{k+\ell+1}f_{\hat{i}_s}(\omega_{\hat{j}_k,\hat{j}_\ell}) & \text{if } i=j_k, j=j_\ell,\\ 0 & \text{if } \{i,j\}\not\subseteq\{j_1,\cdots,j_{n+1}\},\end{cases}$$

so

$$\begin{aligned}&\left(\sum_{i<j}\varepsilon(e_i^*)\varepsilon(e_j^*)\iota([e_i,e_j])\otimes 1\cdot f\right)(\omega)\\ &\qquad=\sum_{k<\ell}\sum_{1\le s\le n}(-1)^{s+1}(-1)^{k+\ell+1}\langle e_{i_s}^*,[e_{j_k},e_{j_\ell}]\rangle f_{\hat{i}_s}(\omega_{\hat{j}_k,\hat{j}_\ell})\\ &\qquad=\sum_{k<\ell}(-1)^{k+\ell+1}f([e_{j_k},e_{j_\ell}]\wedge\omega_{\hat{j}_k,\hat{j}_\ell})\\ &\qquad=\sum_{k<\ell}(-1)^{k+\ell+1}f([e_{j_k},e_{j_\ell}],e_{j_1},\cdots,\hat{e}_{j_k},\cdots,\hat{e}_{j_\ell},\cdots,e_{j_{n+1}}).\end{aligned}$$

Now it is clear that $(\bar{\delta}f)(\omega)=(\delta f)(\omega)$. □

3.2 Semi-infinite structure and semi-infinite cohomology

A Lie (super)algebra L is called *quasi-finite $\mathbb{Z}$-graded* if

$$L=\bigoplus_{n\in\mathbb{Z}}L_n \text{ with } \dim L_n<\infty, \text{ and } [L_n,L_m]\subseteq L_{m+n} \text{ for all } m,n\in\mathbb{Z}.$$

Let

$$L_{\le 0}:=\bigoplus_{n\le 0}L_n \text{ and } L_+:=\bigoplus_{n>0}L_n.$$

The $\mathbb{Z}$-grading on L induces a $\mathbb{Z}_{\le 0}$-grading on $U(L_{\le 0})$, a $\mathbb{Z}_{\ge 0}$-grading on $U(L_+)$ and a $\mathbb{Z}$-grading on $U(L)$, where $U(\mathfrak{a})$ is the universal enveloping algebra of the Lie (super)algebra $\mathfrak{a}$. By the PBW theorem, as $L=L_{\le 0}\oplus L_+$, their universal enveloping algebras, as vector spaces, are related by

$$U(L)\cong U(L_{\le 0})\otimes U(L_+).$$

A typical homogeneous element of $U(L)$ is of the form $\sum_{i=1}^r u_iv_i$ with $u_i\in U(L_{\le 0})$, $v_i\in U(L_+)$ and $\deg(u_iv_i)=\deg(u_jv_j)$ for all i,j.

Definition 3.2.1. Let L be a quasi-finite $\mathbb{Z}$-graded Lie (super)algebra. The *completion* $U(L)^{com}$ of $U(L)$ is the vector space spanned by infinite sums $\sum_{i=-\infty}^{\infty} u_i v_i$ with $u_i \in U(L_{\leq 0}), v_i \in U(L_+)$ such that only a finite number of v_i have degree less than N, i.e., $\sharp\{v_i \mid \deg v_i < N\} < \infty$, for each integer $N \in \mathbb{Z}_{\geq 0}$.

Products are well-defined in the completion, which makes $U(L)^{com}$ into an associative algebra. Obviously, $U(L)$ can be considered as a subalgebra of $U(L)^{com}$.

Definition 3.2.2. Let L be a quasi-finite $\mathbb{Z}$-graded Lie algebra. An L-module M is called *smooth* if for any given $m \in M$, we have $L_n \cdot m = 0$ for $n \gg 0$.

Remark 3.2.3. *One can extend the action of $U(L)$ on a smooth L-module to its completion $U(L)^{com}$. Let M_1, M_2 be smooth modules for L_1, L_2, respectively. Then the tensor product $M_1 \otimes M_2$ is naturally a smooth $L_1 \oplus L_2$-module.*

Definition 3.2.4. Let L_1, L_2 be two associative or Lie superalgebras, and $\varphi : L_1 \to L_2$ be an algebra homomorphism. A *superderivation* of parity $i \in \mathbb{Z}_2$ with respect to φ is a parity-preserving linear map $D : L_1 \to L_2$ satisfying Leibniz's rule

$$D(u \circ_1 v) = D(u) \circ_2 \varphi(v) + (-1)^{i \cdot p(u)} \varphi(u) \circ_2 D(v) \tag{3.4}$$

for all $u, v \in L_1$ with u homogeneous, where $p(u)$ is the parity of u and $\circ_1, \circ_2$ are the multiplications or Lie brackets of L_1, L_2, respectively. We call D even if $i = 0$ and odd if $i = 1$. When one of $\{L_1, L_2\}$ is a Lie superalgebra and the other is an associative superalgebra, we consider both of them as Lie superalgebras.

Remark 3.2.5. *(1) When $L_1 = L_2 = L$ and $\varphi = \mathrm{id}$, D is a superderivation of L.*

(2) A superderivation from a Lie superalgebra L to an associative superalgebra A will induce a same-parity superderivation from $U(L)$ to A.

(3) Let A be an associative superalgebra. Then a superderivation D of A as an associative superalgebra is also a superderivation of A as a Lie superalgebra.

(4) Let L_1 be generated by a subset S. Then a linear map $D : L_1 \to L_2$ satisfying (3.4) for all $u, v \in S$ can be extended uniquely, through Leibniz's rule, to a superderivation from L_1 to L_2, i.e., a superderivation is completely determined by its value on a generating subset.

3.2.1 Semi-infinite structure

Let $L = \bigoplus_{n \in \mathbb{Z}} L_n$ be a quasi-finite $\mathbb{Z}$-graded Lie algebra, with subalgebras $L_{\leq 0} = \bigoplus_{n \leq 0} L_n$ and $L_+ = \bigoplus_{n > 0} L_n$. Let $\{e_i \mid i \leq 0\}$ and $\{e_i \mid i > 0\}$ be homogeneous bases of $L_{\leq 0}$ and L_+, respectively. Homogeneous means that each $e_i \in L_m$ for some $m \in \mathbb{Z}$. We also require that whenever $e_i \in L_m$, we have $e_{i+1} \in L_m$ or $e_{i+1} \in L_{m+1}$. Let $L^* = \bigoplus_{n \in \mathbb{Z}} L_n^*$ be the restricted dual of L with dual basis $\{e_i^* \mid i \in \mathbb{Z}\}$ such that $\langle e_i^*, e_j \rangle = \delta_{i,j}$, where $L_n^* := \mathrm{Hom}_{\mathbb{C}}(L_{-n}, \mathbb{C})$.

Definition 3.2.6. The space $\Lambda^{\infty/2+\bullet}L^*$ of *semi-infinite forms* on L is the vector space spanned by infinite wedge products of L^*, i.e.,

$$\omega = e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots$$

for which there exists an integer $N(\omega)$ such that for all $k > N(\omega)$, we have $i_{k+1} = i_k - 1$.

Let $\iota(L)$ and $\varepsilon(L^*)$ be copies of L and L^*, with bases $\{\iota(e_i) \mid i \in \mathbb{Z}\}$ and $\{\varepsilon(e_i^*) \mid i \in \mathbb{Z}\}$, respectively. For $x \in L$ and $y^* \in L^*$, we denote by $\iota(x)$ and $\varepsilon(y^*)$ the corresponding elements in $\iota(L)$ and $\varepsilon(L^*)$, respectively. Define a Lie superalgebra

$$cl(L) := \iota(L) \oplus \varepsilon(L^*) \oplus \mathbb{C}K$$

with $\iota(L) \oplus \varepsilon(L^*)$ being odd (note that we assume that L is a Lie algebra, hence a purely even space), K being even, and with Lie superbracket: for $x, y \in L$ and $u^*, v^* \in L^*$,

$$[\iota(x), \iota(y)] = [\varepsilon(u^*), \varepsilon(v^*)] = 0, \quad [\iota(x), \varepsilon(u^*)] = \langle u^*, x\rangle K, \quad [K, cl(L)] = 0.$$

Note that $cl(L)$ inherits a natural $\mathbb{Z}$-grading from L with

$$cl(L)_n = \begin{cases} \iota(L_n) \oplus \epsilon(L_n^*) & \text{if } n \neq 0, \\ \iota(L_0) \oplus \epsilon(L_0^*) \oplus \mathbb{C}K & \text{if } n = 0. \end{cases}$$

By the definition of L^*, we have $\iota(e_i) \in cl(L)_n$ and $\varepsilon(e_i^*) \in cl(L)_{-n}$ when $e_i \in L_n$. The Lie superalgebra $cl(L)$ acts on $\Lambda^{\infty/2+\bullet}L^*$ in the following way, K acts as identity, and for $e_{i_0} \in L$,

$$\begin{aligned} \varepsilon(e_{i_0}^*) \cdot e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots &= e_{i_0}^* \wedge e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots , \\ \iota(e_{i_0}) \cdot e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots &= \sum_{k \geq 1} (-1)^{k-1} \langle e_{i_k}^*, e_{i_0}\rangle e_{i_1}^* \wedge \cdots \wedge \hat{e}_{i_k}^* \wedge \cdots . \end{aligned}$$

The Clifford algebra $Cl(L \oplus L^*)$ is defined to be the quotient of $U(cl(L))$ by the ideal generated by $K - 1$, and it also has a well-defined action on $\Lambda^{\infty/2+\bullet}L^*$.

For a subspace V of L, we let $V^\perp = \{w^* \in L^* \mid \langle w^*, u\rangle = 0, \text{ for all } u \in V\}$. Then $L_+^\perp = \bigoplus_{n \geq 0} L_n^*$. Let $\omega_0 = e_0^* \wedge e_{-1}^* \wedge e_{-2}^* \wedge \cdots$. Then

$$\iota(v) \cdot \omega_0 = \varepsilon(u^*) \cdot \omega_0 = 0, \text{ for } v \in L_+ \text{ and } u^* \in L_+^\perp. \tag{3.5}$$

The elements $\iota(v), \varepsilon(u^*)$ with $v \in L_+$ and $u^* \in L_+^\perp$ are called *annihilation operators*. Note that two annihilation operators always anticommute with each other. One can show that the space of semi-infinite forms $\Lambda^{\infty/2+\bullet}L^*$ on L is the irreducible Fock module of $Cl(L \oplus L^*)$ generated by the "vacuum" vector ω_0, with relations defined by (3.5). Every element of $\Lambda^{\infty/2+\bullet}L^*$ can be written as a linear combination of monomials of the form

$$\iota(e_{i_1}) \cdots \iota(e_{i_s}) \varepsilon(e_{j_1}^*) \cdots \varepsilon(e_{j_t}^*) \cdot \omega_0.$$

Remark 3.2.7. *Note that (3.5) implies that $cl(L)_n \cdot \omega_0 = 0$ for $n > 0$. In particular, $\Lambda^{\infty/2+\bullet}L^*$ is a smooth $cl(L)$-module on which K acts as identity, and the action can be extended to $U_1(cl(L))^{com} := U(cl(L))^{com}/(K-1)$.*

We want to define an L-action on $\Lambda^{\infty/2+\bullet}L^*$ through that of $cl(L)$. For the moment we just call it an action, but not necessarily a Lie algebra action. For $x \in L_n$ with $n \neq 0$, we denote by $\rho(x)$, the action of x on $\Lambda^{\infty/2+\bullet}L^*$ defined by

$$\rho(x) \cdot e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots := \sum_{k \geq 1} e_{i_1}^* \wedge \cdots \wedge \mathrm{ad}^* x(e_{i_k}^*) \wedge \cdots , \tag{3.6}$$

where ad^* is the coadjoint action of L on L^*. The above sum is finite, thanks to the definition of semi-infinite forms and the fact that $x \in L_n$ for some $n \neq 0$. It is easy to verify the following relations (as operators on $\Lambda^{\infty/2+\bullet}L^*$): for all $y \in L, z^* \in L^*$,

$$[\rho(x), \iota(y)] = \iota(\mathrm{ad}\, x(y)), \qquad [\rho(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)). \tag{3.7}$$

For $x \in L_0$, we cannot use (3.6) because it may involve an infinite sum. Let $\omega_0 = e_0^* \wedge e_{-1}^* \wedge e_{-2}^* \wedge \cdots$, and choose $\beta \in L_0^*$, considered as a function on L such that $\beta(L_n) = 0$ for all $n \neq 0$. Define $\rho(x) \cdot \omega_0 := \beta(x)\omega_0$, and extend it to an action on $\Lambda^{\infty/2+\bullet}L^*$ by requiring (3.7). This can be done because $\Lambda^{\infty/2+\bullet}L^*$ is irreducible and generated by ω_0 as a module of the Clifford algebra $Cl(L \oplus L^*)$.

To give an explicit expression of the action $\rho(x)$, we define the *normal ordering* of two elements of $\iota(L) \oplus \varepsilon(L^*)$ as follows,

$$: \iota(e_i)\iota(e_j) := \iota(e_i)\iota(e_j), \ : \varepsilon(e_i^*)\varepsilon(e_j^*) := \varepsilon(e_i^*)\varepsilon(e_j^*), \text{ for all } \ i, j \in \mathbb{Z},$$

$$- : \varepsilon(e_j^*)\iota(e_i) : =: \iota(e_i)\varepsilon(e_j^*) := \begin{cases} \iota(e_i)\varepsilon(e_j^*) & \text{if } i \neq j \text{ or } i = j \leq 0, \\ -\varepsilon(e_j^*)\iota(e_i) & \text{if } i = j > 0. \end{cases}$$

Remark 3.2.8. *The idea of normal ordering is to make sure that annihilation operators always appear on the right side of a product. Given a product of multiple operators, for example, $w = \iota(e_{i_1})\varepsilon(e_{j_1}) \cdots \iota(e_{i_s})$, the normal ordering $: w :$ means that we should move the annihilation operators to the right side and then add the sign of the permutation for doing so.*

Thanks to normal ordering, for all $x \in L$, the following elements are well-defined in $U_1(cl(L))^{com}$ and we have

$$\sum_{i \in \mathbb{Z}} : \varepsilon(\mathrm{ad}^* x(e_i^*))\iota(e_i) := \sum_{i \in \mathbb{Z}} : \iota(\mathrm{ad}\, x(e_i))\varepsilon(e_i^*) : .$$

Let

$$\rho^\beta(x) := \sum_{i \in \mathbb{Z}} : \iota(\mathrm{ad}\, x(e_i))\varepsilon(e_i^*) : + \beta(x). \tag{3.8}$$

Then $\rho^\beta(x)$ has a well-defined action on $\Lambda^{\infty/2+\bullet}L^*$ as it is a smooth $cl(L)$-module. Moreover, $\rho^\beta(x)$ satisfies (3.7), i.e., for $y \in L$ and $z^* \in L^*$, we have

$$[\rho^\beta(x), \iota(y)] = \iota(\operatorname{ad} x(y)), \qquad [\rho^\beta(x), \varepsilon(z^*)] = \varepsilon(\operatorname{ad}^* x(z^*)). \tag{3.9}$$

Lemma 3.2.9. *The operator $\rho^\beta(x)$ realizes the action of $\rho(x)$ on $\Lambda^{\infty/2+\bullet}L^*$.*

Proof. Since both $\rho^\beta(x)$ and $\rho(x)$ satisfy (3.7), and $\Lambda^{\infty/2+\bullet}L^*$ is generated by $\omega_0 = e_0^* \wedge e_{-1}^* \wedge e_{-2}^* \wedge \cdots$ as a $Cl(L \oplus L^*)$-module, we only need to show that their actions on ω_0 coincide. For simplicity, we assume that $x = e_{i_x}$. By definition

$$\rho(e_{i_x}) \cdot \omega_0 = \begin{cases} \beta(e_{i_x})\omega_0 & \text{if } e_{i_x} \in L_0, \\ \sum_{k \geq 0} e_0^* \wedge \cdots \wedge \operatorname{ad}^* e_{i_x}(e_{-k}^*) \wedge \cdots & \text{if } e_{i_x} \in L_n \text{ and } n \neq 0. \end{cases}$$

Now let us calculate the action of $\rho^\beta(e_{i_x})$ on ω_0. When $e_{i_x} \in L_0$, there is an annihilation operator in each summand : $\iota(\operatorname{ad} e_{i_x}(e_i))\varepsilon(e_i^*)$: since $[L_0, L_n] \subseteq L_n$. Therefore, the sum $\sum_i$: $\iota(\operatorname{ad} e_{i_x}(e_i))\varepsilon(e_i^*)$: acts as zero on ω_0 and $\rho^\beta(e_{i_x}) \cdot \omega_0 = \beta(e_{i_x})\omega_0$. When $e_{i_x} \in L_n$ for some $n \neq 0$, we have $\beta(e_{i_x}) = 0$. Moreover, $\varepsilon(\operatorname{ad}^* e_{i_x}(e_i^*))$ always anticommutes with $\iota(e_i)$ as $[L_n, L_m] \subseteq L_{m+n}$, so we can drop :: in $\rho^\beta(e_{i_x})$. Remember that $\iota(e_i) \cdot \omega_0 = 0$ for all $i > 0$, so

$$\begin{aligned} \rho^\beta(e_{i_x}) \cdot \omega_0 &= \sum_{i \leq 0} \varepsilon(\operatorname{ad}^* e_{i_x}(e_i^*)) \cdot (-1)^i e_0^* \wedge \cdots \wedge \hat{e}_i^* \wedge \cdots \\ &= \sum_{i \leq 0} e_0^* \wedge \cdots \wedge \operatorname{ad}^* e_{i_x}(e_i^*) \wedge \cdots . \end{aligned}$$

□

One can show that the centers of the Clifford algebra $Cl(L \oplus L^*)$ and its completion $U_1(cl(L))^{com}$ are both trivial, i.e., they only contain the constants.

For $x, y \in L$, define

$$\gamma^\beta(x, y) := [\rho^\beta(x), \rho^\beta(y)] - \rho^\beta([x, y]). \tag{3.10}$$

It is clear that $\Lambda^{\infty/2+\bullet}L^*$ admits an L-module structure under $\rho^\beta(x)$ if and only if $\gamma^\beta(x, y) = 0$ for all $x, y \in L$. One can show that $\gamma^\beta(x, y)$ is central hence a constant in $U_1(cl(L))^{com}$. Indeed, it is a 2-cocycle, i.e.,

$$\gamma^\beta(x, [y, z]) + \gamma^\beta(y, [z, x]) + \gamma^\beta(z, [y, x]) = 0 \text{ for all } \quad x, y, z \in L.$$

Moreover, one can show that $\gamma^\beta(L_m, L_n) = 0$ whenever $m + n \neq 0$ [Vor93].

Definition 3.2.10. We say that L admits a *semi-infinite structure* through ρ^β if $\gamma^\beta(\cdot, \cdot) \equiv 0$, i.e., if $\Lambda^{\infty/2+\bullet}L^*$ is an L-module under the action $\rho^\beta(x)$. We say that L admits a semi-infinite structure if L admits a semi-infinite structure through ρ^β for some $\beta \in L_0^*$.

Remark 3.2.11. *We can drop the restriction that $\beta \in L_0^*$ for a more general definition. In Chapter 4, when we realize affine W-algebras as semi-infinite cohomology, we are in the more general case. But for the existence of a semi-infinite structure, the part which belongs to L_0^* is essential. For example, let $\beta = \sum_i \beta_i \in L^*$ with $\beta_i \in L_i^*$. Then ρ^β gives L a semi-infinite structure if and only if ρ^{β_0} does and $\partial\beta_i = 0$ for all $i \neq 0$. Here $\partial\beta_i(x,y) := \beta_i([x,y])$ for $x, y \in L$.*

Example 3.2.12. If L is abelian, it always admits a semi-infinite structure. When $H^2(L,\mathbb{C}) = 0$, every 2-cocycle is a coboundary. If $\gamma^\beta(\cdot,\cdot) \neq 0$, we can choose some $\beta' \in L^*$ (by [Vor93], we can choose $\beta' \in L_0^*$), such that $\partial\beta' = \gamma^\beta(\cdot,\cdot)$, then $\rho^{\beta-\beta'}$ gives a semi-infinite structure for L. For example, affine Kac-Moody algebras and the Virasoro algebra admit semi-infinite structures.

Let $\mathfrak{a}$ be a finite-dimensional Lie algebra. Recall that the affinization of $\mathfrak{a}$ is the tensor product $\hat{\mathfrak{a}} := \mathfrak{a} \otimes \mathbb{C}[t,t^{-1}]$ with Lie bracket: $[a \otimes t^n, b \otimes t^m] = [a,b] \otimes t^{m+n}$ for all $a, b \in \mathfrak{a}$ and $m, n \in \mathbb{Z}$, where $\mathbb{C}[t,t^{-1}]$ is the ring of Laurent polynomials. It has a natural $\mathbb{Z}$-grading with $\hat{\mathfrak{a}}_n := \mathfrak{a} \otimes t^n$.

Proposition 3.2.13. *Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra. Then $\hat{\mathfrak{n}}$ admits a semi-infinite structure.*

Proof. Let $\dim \mathfrak{n} = d$ and $\mathcal{B} := \{e_i\}_{1\le i\le d}$ be a basis of $\mathfrak{n}$, with structure constants $\{c_{i,j}^k\}$ such that $[e_i, e_j] = \sum_{k=1}^d c_{i,j}^k e_k$. Since $\mathfrak{n}$ is nilpotent, by Engel's theorem, we can choose the basis $\mathcal{B}$, such that $c_{i,j}^k = 0$ for $k \ge j$. In the language of matrices, $\operatorname{ad} e_i \in \mathfrak{gl}(\mathfrak{n})$ with respect to $\mathcal{B}$ are strictly upper triangular matrices for all i. In particular, we have $c_{i,j}^j = 0$. We fix such a basis $\mathcal{B}$, and let $\{e_i^*\}_{1\le i\le d}$ be the dual basis of $\mathfrak{n}^*$. Identify the restricted dual $\hat{\mathfrak{n}}^*$ of $\hat{\mathfrak{n}}$ with $\mathfrak{n}^* \otimes \mathbb{C}[t,t^{-1}]$ through the pairing $\langle e_j^* \otimes t^m, e_i \otimes t^n\rangle = \delta_{n,-m}\delta_{i,j}$. For convenience, we denote by $e_{i,n} := e_i \otimes t^n$ and $e_{i,n}^* := e_i^* \otimes t^{-n}$. Then $\{e_{i,n}\}$ and $\{e_{i,n}^*\}$ form dual bases of $\hat{\mathfrak{n}}$ and $\hat{\mathfrak{n}}^*$, respectively. The adjoint action gives

$$\operatorname{ad} e_{i,n}(e_{j,m}) = [e_{i,n}, e_{j,m}] = [e_i, e_j] \otimes t^{m+n} = \sum_{k=1}^{d} c_{i,j}^k e_{k,m+n}.$$

For the coadjoint action, we have $\operatorname{ad}^* e_{i,n}(e_{j,m}^*) = \sum_{k=1}^d c_{k,i}^j e_{k,m-n}^*$.

Let

$$\rho^0(x) = \sum_{\substack{i=1,\cdots,d,\\ n\in\mathbb{Z}}} :\iota(\operatorname{ad} x(e_{i,n}))\varepsilon(e_{i,n}^*):.$$

We show that $\gamma^0(x,y) := [\rho^0(x), \rho^0(y)] - \rho^0([x,y]) = 0$ for all $x, y \in \hat{\mathfrak{n}}$, i.e., $\hat{\mathfrak{n}}$ admits a semi-infinite structure through ρ^0.

For simplicity, assume that $x = e_{i_x,n_x}$ and $y = e_{i_y,n_y}$. Since $\iota(\operatorname{ad} x(e_{i,n}))$ anticommutes with $\varepsilon(e_{i,n}^*)$

by the choice of basis of $\mathfrak{n}$, we can drop the normal ordering :: in $\rho^0(x)$, so we have

$$
\begin{aligned}
&[\rho^0(e_{i_x,n_x}), \rho^0(e_{i_y,n_y})] \\
&\qquad = \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \left[\iota(\operatorname{ad} e_{i_x,n_x}(e_{i,n}))\varepsilon(e^*_{i,n}),\ \varepsilon(\operatorname{ad}^* e_{i_y,n_y}(e^*_{j,m}))\iota(e_{j,m})\right] \\
&\qquad = A + B,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \iota(\operatorname{ad} e_{i_x,n_x}(e_{i,n})) \left[\varepsilon(e^*_{i,n}), \varepsilon(\operatorname{ad}^* e_{i_y,n_y}(e^*_{j,m}))\iota(e_{j,m})\right], \\
B &= \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \left[\iota(\operatorname{ad} e_{i_x,n_x}(e_{i,n})), \varepsilon(\operatorname{ad}^* e_{i_y,n_y}(e^*_{j,m}))\iota(e_{j,m})\right] \varepsilon(e^*_{i,n}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
A &= - \sum_{\substack{i=1,\cdots,d,\\ n\in\mathbb{Z}}} \iota(\operatorname{ad} e_{i_x,n_x}(e_{i,n}))\varepsilon(\operatorname{ad}^* e_{i_y,n_y}(e^*_{i,n})) \\
&= - \sum_{\substack{i,j,k=1,\cdots,d,\\ n\in\mathbb{Z}}} c^j_{i_x,i} c^i_{k,i_y} \iota(e_{j,n+n_x})\varepsilon(e^*_{k,n-n_y}),
\end{aligned}
$$

and

$$
\begin{aligned}
B &= \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \langle \operatorname{ad}^* e_{i_y,n_y}(e^*_{j,m}), \operatorname{ad} e_{i_x,n_x}(e_{i,n})\rangle \iota(e_{j,m})\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ n\in\mathbb{Z}}} c^j_{k,i_y} c^k_{i_x,i} \iota(e_{j,n+n_x+n_y})\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ m\in\mathbb{Z}}} c^j_{k,i_y} c^k_{i_x,i} \iota(e_{j,m+n_x})\varepsilon(e^*_{i,m-n_y}).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\rho^0([e_{i_x,n_x}, e_{i_y,n_y}]) &= \sum_{\substack{i=1,\cdots,d,\\ n\in\mathbb{Z}}} \iota(\operatorname{ad}[e_{i_x,n_x}, e_{i_y,n_y}](e_{i,n}))\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ n\in\mathbb{Z}}} c^j_{i_x,i_y} c^k_{j,i} \iota(e_{k,n+n_x+n_y})\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ m\in\mathbb{Z}}} c^j_{i_x,i_y} c^k_{j,i} \iota(e_{k,m+n_x})\varepsilon(e^*_{i,m-n_y}).
\end{aligned}
$$

Now $[\rho^0(x), \rho^0(y)] - \rho^0([x,y]) = 0$ comes from the Jacobi identity of the structure constants,

$$
- \sum_i c^j_{i_x,i} c^i_{k,i_y} + \sum_i c^j_{i,i_y} c^i_{i_x,k} = \sum_i c^i_{i_x,i_y} c^j_{i,k}.
$$

□

3.2.2 Semi-infinite cohomology

In this subsection, we assume that L is a quasi-finite $\mathbb{Z}$-graded Lie algebra admitting a semi-infinite structure through ρ^β defined by (3.8), i.e., $\gamma^\beta(\cdot,\cdot) \equiv 0$ and the map $\rho^\beta : L \to U_1(cl(L))^{com}$ defined by $x \mapsto \rho^\beta(x)$ is a Lie algebra homomorphism, which gives $\Lambda^{\infty/2+\bullet}L^*$ an L-module structure.

Let $\theta^\beta : L \to U(L) \otimes U_1(cl(L))^{com}$ be the map defined by

$$\theta^\beta(x) := x + \rho^\beta(x). \tag{3.11}$$

Remark 3.2.14. *Note that we omitted the tensor product $\otimes$ in (3.11), so $\theta^\beta(x) = x \otimes 1 + 1 \otimes \rho^\beta(x)$. We will use the same notation in the sequel.*

The map θ^β is obviously a Lie algebra homomorphism. Let M be a smooth L-module. Then the tensor product $M \otimes \Lambda^{\infty/2+\bullet}L^*$ is naturally a $U(L)\otimes U_1(cl(L))^{com}$-module hence a smooth L-module under the action $\theta^\beta(x)$. Since x commutes with $\iota(L)$ and $\varepsilon(L^*)$, we have: for all $y \in L, z^* \in L^*$,

$$[\theta^\beta(x), \iota(y)] = \iota([x,y]), \qquad [\theta^\beta(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)).$$

Let

$$\begin{aligned} d^\beta &= \sum_{i\in\mathbb{Z}} e_i \varepsilon(e_i^*) - \sum_{i<j} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta) \\ &= \sum_{i\in\mathbb{Z}} e_i \varepsilon(e_i^*) - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta). \end{aligned} \tag{3.12}$$

Then $d^\beta \in U(L)^{com} \otimes U_1(cl(L))^{com}$ has a well-defined action on $M \otimes \Lambda^{\infty/2+\bullet}L^*$.

Lemma 3.2.15. *We have $[d^\beta, \iota(x)] = \theta^\beta(x)$ for all $x \in L$.*

Proof. For simplicity, we assume that $x = e_k$ for some $k \in \mathbb{Z}$. Then

$$\left[\sum_{i\in\mathbb{Z}} e_i\varepsilon(e_i^*) + \varepsilon(\beta), \iota(e_k)\right] = e_k + \beta(e_k),$$

and

$$\begin{aligned} &\left[-\sum_{i<j} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*):, \iota(e_k)\right] \\ &\qquad = -\sum_{i<k} :\iota([e_i,e_k])\varepsilon(e_i^*): + \sum_{k<j} :\iota([e_k,e_j])\varepsilon(e_j^*): \\ &\qquad = \sum_{i\in\mathbb{Z}} :\iota(\mathrm{ad}\, e_k(e_i))\varepsilon(e_i^*):. \end{aligned}$$

Therefore, we have $[d^\beta, \iota(e_k)] = \theta^\beta(e_k)$. ∎

We define a charge grading on $cl(L)$ by setting

$$-\text{cdeg}\,\iota(x) = \text{cdeg}\,\varepsilon(y^*) = 1 \quad \text{for } x \in L, y^* \in L^*, \quad \text{and} \quad \text{cdeg}\,K = 0. \tag{3.13}$$

When we refer to the charge gradation, we will add the superscript $^{\circledast}$. We have

$$cl(L) = cl(L)^{\circledast}_{-1} \oplus cl(L)^{\circledast}_{0} \oplus cl(L)^{\circledast}_{1}$$

with $cl(L)^{\circledast}_1 = \varepsilon(L^*), cl(L)^{\circledast}_0 = \mathbb{C}K$ and $cl(L)^{\circledast}_{-1} = \iota(L)$. This induces a charge gradation on $U(cl(L))$ and also on the Clifford algebra $Cl(L \oplus L^*)$. As a simple module of $Cl(L \oplus L^*)$, the space of semi-infinite forms $\Lambda^{\infty/2+\bullet}L^*$ inherits a charge gradation if we set $\text{cdeg}\,\omega_0 = 0$, with

$$\Lambda^{\infty/2+n}L^* := (\Lambda^{\infty/2+\bullet}L^*)^{\circledast}_n = \text{span}_{\mathbb{C}}\{\iota(e_{i_1}) \cdots \iota(e_{i_s})\varepsilon(e^*_{j_1}) \cdots \varepsilon(e^*_{j_t}) \cdot \omega_0 \mid t - s = n\}.$$

With respect to the charge gradation, the operator $\rho^\beta(x)$ is of degree zero for all $x \in L$, so each component $\Lambda^{\infty/2+n}L^*$ is an L-submodule. If we define the charge degree of M to be zero, then d^β is a charge degree 1 operator on $M \otimes \Lambda^{\infty/2+\bullet}L^*$.

Proposition 3.2.16 ([Vor93], Proposition 2.6)**.** *The operator d^β does not depend on the choice of basis of L, and $(d^\beta)^2 = 0$.*

Definition 3.2.17. The complex $(M \otimes \Lambda^{\infty/2+\bullet}L^*, d^\beta)$ is called the *Feigin standard complex* and its cohomology $H^{\infty/2+\bullet}(L, \beta, M)$ the *semi-infinite cohomology* of L with coefficients in M. When $\beta = 0$, we write just as $H^{\infty/2+\bullet}(L, M)$.

Remark 3.2.18. *There is an interesting characterization of the differential d^β in [Akm93] and in [Ara17] for affine W-algebras in the principal nilpotent cases, which can be realized as a semi-infinite cohomology. To contrast with our adjusted version in the next section, we will also call the cohomology in Definition 3.2.17 ordinary semi-infinite cohomology.*

We write β in the cohomology because it plays some role. Indeed, if $\rho^{\beta'}$ gives another semi-infinite structure, one can show that $(\beta - \beta')([L, L]) = 0$, so $\beta - \beta'$ defines a 1-dimensional module $\mathbb{C}_{\beta-\beta'}$ of L, on which $x \in L$ acts as $(\beta - \beta')(x)$.

Proposition 3.2.19 ([Vor93], Proposition 2.7)**.** *If both ρ^β and $\rho^{\beta'}$ give semi-infinite structures on L, then*

$$H^{\infty/2+\bullet}(L, \beta, M) \cong H^{\infty/2+\bullet}(L, \beta', M \otimes \mathbb{C}_{\beta-\beta'}).$$

3.3 An adjustment when the 2-cocycle $\gamma^\beta(\cdot, \cdot)$ is not identically zero

Recall the notation in the previous section. We assume that $\gamma^\beta(\cdot, \cdot)$ is not identically zero in this section, i.e., ρ^β does not give a semi-infinite structure on L.

3.3.1 What is the problem

Let d^β be the operator defined by (3.12) and let us consider the value $[[(d^\beta)^2, \iota(x)], \iota(y)]$ for $x, y \in L$. Since d^β is odd, we have $(d^\beta)^2 = \dfrac{1}{2}[d^\beta, d^\beta]$, hence $[(d^\beta)^2, \iota(x)] = [d^\beta, [d^\beta, \iota(x)]]$. By Lemma 3.2.15, we have $[d^\beta, \iota(x)] = \theta^\beta(x)$ (though we assume that $\gamma^\beta(\cdot,\cdot) \equiv 0$ in that section, the calculations in Lemma 3.2.15 still hold), so

$$\begin{aligned} [[(d^\beta)^2, \iota(x)], \iota(y)] &= [[d^\beta, \theta^\beta(x)], \iota(y)] \\ &= [d^\beta, [\theta^\beta(x), \iota(y)]] + [[d^\beta, \iota(y)], \theta^\beta(x)] \\ &= [d^\beta, \iota([x, y])] + [\theta^\beta(y), \theta^\beta(x)] \\ &= \theta^\beta([x, y]) - [\theta^\beta(x), \theta^\beta(y)] \\ &= -\gamma^\beta(x, y). \end{aligned} \tag{3.14}$$

In particular, the operator d^β is not of square zero if $\gamma^\beta(\cdot,\cdot)$ is not identically zero.

Let $\ker\,\gamma^\beta := \{x \in L \mid \gamma(x, L) \equiv 0\}$ be the radical of the 2-cocycle $\gamma^\beta(\cdot,\cdot)$. Then $\ker\,\gamma^\beta$ is obviously a graded subalgebra of L. Let us choose a graded complement of $\ker\,\gamma^\beta$ in L, which we denote by F_β. Then $L = \ker\,\gamma^\beta \oplus F_\beta$, and $\gamma^\beta(\cdot,\cdot)$ is non-degenerate on F_β. Let $\epsilon(F_\beta)$ be a copy of F_β. For $x \in L$, we use $\epsilon(x)$ to denote its projection in F_β but considered as an element of $\epsilon(F_\beta)$. Then $\epsilon(\ker\,\gamma^\beta) = 0$.

Consider the Lie superalgebra

$$c(L) := \iota(L) \oplus \varepsilon(L^*) \oplus \mathbb{C}K \oplus \epsilon(F_\beta),$$

which contains $cl(L)$ as a subalgebra. By definition, the subspace $\epsilon(F_\beta)$ is even, commutes with $cl(L)$, and has bracket: $[\epsilon(x), \epsilon(y)] = -\gamma^\beta(x, y)K$ for $x, y \in F_\beta$. Since F_β is a graded subspace of L, the subalgebra $\epsilon(F_\beta) \oplus \mathbb{C}K$ is $\mathbb{Z}$-graded with

$$(\epsilon(F_\beta) \oplus \mathbb{C}K)_n = \begin{cases} \epsilon((F_\beta)_n) & \text{if } n \neq 0, \\ \epsilon((F_\beta)_0) \oplus \mathbb{C}K & \text{if } n = 0. \end{cases}$$

The subspace $\epsilon(F_\beta)_+ := \left(\bigoplus_{n>0} \epsilon(F_\beta)_n\right) \oplus \mathbb{C}K$ is an abelian subalgebra, thanks to the property that $\gamma^\beta(L_m, L_n) \equiv 0$ if $m + n \neq 0$. Let $\mathbb{C}$ be the 1-dimensional module of this abelian subalgebra on which $\bigoplus_{n>0} \epsilon(F_\beta)_n$ acts as zero and K acts as the identity. We call the induced module

$$\mathfrak{F}_\beta = \operatorname{Ind}_{\epsilon(F_\beta)_+}^{\epsilon(F_\beta)\oplus\mathbb{C}K} \mathbb{C} \tag{3.15}$$

the *Fock representation* of $\epsilon(F_\beta) \oplus \mathbb{C}K$, which is obviously smooth. Remember that $\Lambda^{\infty/2+\bullet}L^*$ is a smooth $cl(L)$-module on which K also acts as identity, so $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ is a smooth $c(L)$-module.

Let $U_1(c(L))^{com} := U(c(L))^{com}/(K-1)$, and define a map $\bar{\rho}^\beta : L \to U_1(c(L))^{com}$ by

$$\bar{\rho}^\beta(x) := \rho^\beta(x) + \epsilon(x).$$

Then $\bar{\rho}^\beta(x)$ has a well-defined action on $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$, and for $x, y \in L$, $z^* \in L^*$, we have

$$[\bar{\rho}^\beta(x), \iota(y)] = \iota([x,y]), \quad [\bar{\rho}^\beta(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)), \quad [\bar{\rho}^\beta(x), \epsilon(y)] = -\gamma^\beta(x,y). \tag{3.16}$$

Let $s(L) = L \oplus c(L)$ be the direct sum of L and $c(L)$. Then $s(L)$ inherits a natural $\mathbb{Z}$-grading from L and $c(L)$. Let

$$U_1(s(L))^{com} := U(s(L))^{com}/(K-1) \cong U(L)^{com} \otimes U_1(c(L))^{com},$$

and

$$\bar{\theta}^\beta(x) = x + \bar{\rho}^\beta(x) \in U_1(s(L))^{com}. \tag{3.17}$$

Let M be a smooth L-module. Then $\bar{\theta}^\beta(x)$ has a well-defined action on $M \otimes \Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$. We have $[\bar{\theta}^\beta(x), y] = [x,y]$ for all $x, y \in L$, moreover,

$$[\bar{\theta}^\beta(x), \iota(y)] = \iota([x,y]), \quad [\bar{\theta}^\beta(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)), \quad [\bar{\theta}^\beta(x), \epsilon(y)] = -\gamma^\beta(x,y). \tag{3.18}$$

Lemma 3.3.1. *The map $\bar{\rho}^\beta : L \longrightarrow U_1(c(L))^{com}$ is a Lie algebra homomorphism if $[L,L] \subseteq \ker \gamma^\beta$.*

Proof. We need to prove $\bar{\rho}^\beta([x,y]) = [\bar{\rho}^\beta(x), \bar{\rho}^\beta(y)]$ for all $x, y \in L$. But we have

$$\begin{aligned}
[\bar{\rho}^\beta(x), \bar{\rho}^\beta(y)] &= [\rho^\beta(x) + \epsilon(x), \rho^\beta(y) + \epsilon(y)] \\
&= [\rho^\beta(x), \rho^\beta(y)] + [\epsilon(x), \epsilon(y)] \\
&= \rho^\beta([x,y]) + \gamma^\beta(x,y) - \gamma^\beta(x,y) \\
&= \rho^\beta([x,y])
\end{aligned}$$

and $\bar{\rho}^\beta([x,y]) = \rho^\beta([x,y])$ if $\epsilon([x,y]) \equiv 0$, i.e., if $[L,L] \subseteq \ker \gamma^\beta$. ☐

Remark 3.3.2. *Lemma 3.3.1 tells us that even though $\Lambda^{\infty/2+\bullet}L^*$ is not an L-module under the action $\rho^\beta(x)$, the tensor product $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ is under $\bar{\rho}^\beta(x)$.*

Assumption: From now on, we assume that $[L,L] \subseteq \ker \gamma^\beta$ is satisfied.

3.3.2 Construction and characterization of a square zero differential

We extend the charge gradation (see (3.13)) on $cl(L)$ to $c(L)$ by setting $\mathrm{cdeg}\, \epsilon(F_\beta) = 0$, and then to $s(L)$ by setting $\mathrm{cdeg}\, L = 0$. As usual, we denote the charge gradation by adding a superscript *. These charge gradations induce another $\mathbb{Z}$-grading on their universal enveloping algebras, which are different from those induced from the quasi-finite $\mathbb{Z}$-grading. At the module level, if we set $\mathrm{cdeg}\, M = \mathrm{cdeg}\, \mathfrak{F}_\beta = 0$ for a smooth L-module M, and the charge gradation on $\Lambda^{\infty/2+\bullet}L^*$ as before, then $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ is a $\mathbb{Z}$-graded $c(L)$-module and $M \otimes \Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ a $\mathbb{Z}$-graded $s(L)$-module under the charge gradations.

Let $i_c : c(L) \hookrightarrow U_1(c(L))^{com}$ and $i_s : s(L) \hookrightarrow U_1(s(L))^{com}$ be the canonical inclusions.

Definition 3.3.3. A superderivation D with respect to i_c or i_s, is said to be of *charge degree* N if $D(c(L)^{\circledast}_n) \subseteq U_1(c(L))^{com,\circledast}_{n+N}$ or $D(s(L)^{\circledast}_n) \subseteq U_1(s(L))^{com,\circledast}_{n+N}$, respectively. A superderivation D of $c(L)$ or of $s(L)$ is said to be of *charge degree* N if $D(c(L)^{\circledast}_n) \subseteq c(L)^{\circledast}_{n+N}$ or $D(s(L)^{\circledast}_n) \subseteq s(L)^{\circledast}_{n+N}$, respectively.

Define an action of L on $c(L)$ as follows. For $x, y \in L, z \in L^*$,

$$x \cdot \iota(y) = \iota([x,y]), \quad x \cdot \varepsilon(z^*) = \varepsilon(\mathrm{ad}^* x(z^*)), \quad x \cdot \epsilon(y) = -\gamma^\beta(x,y)K, \quad x \cdot K = 0.$$

We extend this action to $s(L)$ by letting L act on itself by the adjoint action.

Lemma 3.3.4. *The actions of $x \in L$ on $c(L)$ and $s(L)$ are even derivations of charge degree zero.*

Proof. This can be verified by direct calculations, as we know explicitly both the Lie brackets of $c(L), s(L)$ and the actions of L on them. They are obviously of charge degree zero. □

Remark 3.3.5. *The actions of x on $c(L)$ and $s(L)$ induce even derivations of charge degree zero on $U_1(c(L))^{com}$ and $U_1(s(L))^{com}$, respectively. The inner derivations $[\bar{\rho}^\beta(x), \cdot]$ and $[\bar{\theta}^\beta(x), \cdot]$ realize the actions of x on $U_1(c(L))^{com}$ and $U_1(s(L))^{com}$, respectively, by (3.16) and (3.18).*

Lemma 3.3.6. *Let $u \in U_1(s(L))^{com}$ be a charge degree ≥ 1 element. Then $[u, \iota(x)] = 0$ for all $x \in L$ only if $u = 0$.*

Proof. As $\mathrm{cdeg}\, u \geq 1$, if u is not zero, we can write

$$u = w\varepsilon(e_k^*) + v \qquad \text{or} \qquad u = \varepsilon(e_k^*)w + v$$

for some $k \in \mathbb{Z}$ with $w, v \in U_1(s(L))^{com}$ and $w \neq 0$, such that $\varepsilon(e_k^*)$ does not appear in w or v, i.e.,

$$[w, \iota(e_k)] = [v, \iota(e_k)] = 0.$$

Then $[u, \iota(e_k)] = w \neq 0$ gives a contradiction. □

Lemma 3.3.7. *Let D be a superderivation of charge degree ≥ 1 with respect to $i_s : s(L) \hookrightarrow U_1(s(L))^{com}$, and suppose that D kills K. Then D is determined by its value on $\iota(L)$.*

Proof. Since $s(L)$ is generated by $L \oplus \iota(L) \oplus \varepsilon(L^*) \oplus \epsilon(F_\beta)$, we just need to show that the value of D on $L \oplus \varepsilon(L^*) \oplus \epsilon(F_\beta)$ is determined by its value on $\iota(L)$. Let D' be another superderivation, such that D' kills K and coincide with D on $\iota(L)$. We show that $D = D'$. Since $D - D'$ is also a superderivation, we have

$$(D - D')[u,v] = [(D - D')u, v] + (-1)^{i \cdot p(u)}[u, (D - D')v] \tag{3.19}$$

for all $u, v \in s(L)$, where i is the parity of D and D'.

Note that $[s(L), \iota(L)] \subseteq \mathbb{C}K$ and $(D - D')K = (D - D')\iota(L) = 0$. Let $u \in s(L), v = \iota(x) \in \iota(L)$ in (3.19). Then we have

$$[(D - D')u, \iota(x)] = 0. \tag{3.20}$$

If $u \in \iota(L)$, then $(D - D')u = 0$. If $u \in L \oplus \epsilon(F_\beta) \oplus \varepsilon(L^*)$, then note that $\operatorname{cdeg}(D - D')u \geq 1$ when $u \in L \oplus \epsilon(F_\beta)$, and $\operatorname{cdeg}(D - D')u \geq 2$ when $u \in \varepsilon(L^*)$. Since (3.20) holds for all $\iota(x) \in \iota(L)$, Lemma 3.3.6 ensures that $(D - D')u = 0$, i.e., $D = D'$ on $s(L)$. □

Remark 3.3.8. *An equivalent statement of Lemma 3.3.7 is, given a charge degree ≥ 1 superderivation with respect to the inclusion $i_{\iota(L)} : \iota(L) \to U_1(s(L))^{com}$, we can extend it to be a superderivation of the same charge degree with respect to the inclusion $i_s : s(L) \to U_1(s(L))^{com}$ in a unique way.*

Recall that $\bar{\theta}^\beta(x)$ defined by (3.17) is even and satisfies (3.18), in particular, we have

$$[\bar{\theta}^\beta(x), \iota(y)] - [\iota(x), \bar{\theta}^\beta(y)] = \iota([x, y]) + \iota([y, x]) = 0.$$

As $\iota(L)$ is an abelian subalgebra of $s(L)$, the map $D : \iota(L) \to U_1(s(L))^{com}$ sending $\iota(x)$ to $\bar{\theta}^\beta(x)$ is an odd superderivation of charge degree 1 with respect to $i_{\iota(L)}$, so it can be extended to be a superderivation with respect to i_s in a unique way.

Let

$$\bar{d}^\beta = d^\beta + \sum_{i \in \mathbb{Z}} \varepsilon(e_i^*)\epsilon(e_i). \tag{3.21}$$

Theorem 3.3.9. *The element $\bar{d}^\beta$ defined by (3.21) is the unique element in $U_1(s(L))^{com}$ of charge degree 1, such that $[\bar{d}^\beta, \iota(x)] = \bar{\theta}^\beta(x)$ for all $x \in L$, and we have $(\bar{d}^\beta)^2 = 0$.*

Proof. By Lemma 3.2.15, we already have $[d^\beta, \iota(x)] = \theta^\beta(x)$, so we only need to show that

$$\sum_{i \in \mathbb{Z}} [\varepsilon(e_i^*)\epsilon(e_i), \iota(x)] = \epsilon(x).$$

This is obvious for $x = e_k$ hence true for all $x \in L$. The uniqueness is by Lemma 3.3.6.

The operators $[(\bar{d}^\beta)^2, \cdot]$ and $[[(\bar{d}^\beta)^2, \iota(x)], \cdot]$ are derivations of charge degree 2 and 1, respectively, if they are non-zero. By Lemma 3.3.7, they are completely determined by their value on $\iota(L)$. Recall the calculations in (3.14). Since $[L, L] \subseteq \ker \gamma^\beta$ and $[\bar{d}^\beta, \iota(x)] = \bar{\theta}^\beta(x)$, we have

$$\begin{aligned}
[[(\bar{d}^\beta)^2, \iota(x)], \iota(y)] &= \bar{\theta}^\beta([x, y]) - [\bar{\theta}^\beta(x), \bar{\theta}^\beta(y)] \\
&= \rho^\beta([x, y]) + [x, y] - [\rho^\beta(x) + x + \epsilon(x), \rho^\beta(y) + y + \epsilon(y)] \\
&= \rho^\beta([x, y]) - [\rho^\beta(x), \rho^\beta(y)] + \gamma^\beta(x, y) \\
&= 0,
\end{aligned}$$

for $x, y \in L$. Lemma 3.3.6 then implies that $[(\bar{d}^\beta)^2, \iota(x)] = 0$ for all $x \in L$ hence $(\bar{d}^\beta)^2 = 0$. □

Definition 3.3.10. We call the complex $(M \otimes \Lambda^{\infty/2+\bullet} L^* \otimes \mathfrak{F}_\beta, \bar{d}^\beta)$ the *adjusted Feigin complex* with respect to β, and its cohomology $H_a^{\infty/2+\bullet}(L, \beta, M)$ the *adjusted semi-infinite cohomology* of L with coefficients in M, with respect to β.

Remark 3.3.11. *Note that we used a subscript "a" in the adjusted semi-infinite cohomology.*

3.3.3 Comparison with ordinary semi-infinite cohomology

Our adjustment sometimes gives nothing new but ordinary semi-infinite cohomology with coefficients in another module. Assume that $\rho^\beta(x)$ gives a semi-infinite structure on L, and $\beta' \in \bigoplus_{n\geq 0} L_n^*$ is a 1-cochain[1] such that $\partial\beta' \neq 0$ but $\partial\beta'([L,L],L) = 0$, where $\partial\beta'(x,y) = \beta'([x,y])$. Then $\gamma^{\beta+\beta'} = -\partial\beta' \neq 0$ and $[L,L] \subseteq \ker \gamma^{\beta+\beta'}$. We can therefore talk about adjusted semi-infinite cohomology of L with coefficients in a smooth module M with respect to $\beta+\beta'$, which is the cohomology of the complex $(M \otimes \Lambda^{\infty/2+\bullet} \otimes \mathfrak{F}_{\beta+\beta'}, \bar{d}^{\beta+\beta'})$.

Recall that

$$\begin{aligned}\bar{d}^{\beta+\beta'} &= \sum_{i\in\mathbb{Z}} e_i \varepsilon(e_i^*) - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta+\beta') + \sum_{i\in\mathbb{Z}} \varepsilon(e_i^*)\epsilon(e_i) \\ &= \sum_{i\in\mathbb{Z}} \varepsilon(e_i^*)(e_i + \beta'(e_i) + \epsilon(e_i)) - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta),\end{aligned}$$

and

$$[\bar{d}^{\beta+\beta'}, \iota(x)] = x + \beta'(x) + \epsilon(x) + \rho^\beta(x).$$

On the other hand, since $[\epsilon(x), \epsilon(y)] = -\gamma^{\beta+\beta'}(x,y) = \beta'([x,y])$ and $\epsilon([x,y]) \equiv 0$, we have

$$[x + \beta'(x) + \epsilon(x), y + \beta'(y) + \epsilon(y)] = [x,y] + \beta'([x,y]),$$

that is, $M \otimes \mathfrak{F}_{\beta+\beta'}$ is an L-module under the action $x + \beta'(x) + \epsilon(x)$, and it is smooth. Therefore, we have the following theorem.

Theorem 3.3.12. *Let β, β' be as above. Then*

$$H_a^{\infty/2+\bullet}(L, \beta+\beta', M) \cong H^{\infty/2+\bullet}(L, \beta, M \otimes \mathfrak{F}_{\beta+\beta'}).$$

[1] We require that $\beta' \in \bigoplus_{n\geq 0} L_n^*$ to make sure that in the construction of $\mathfrak{F}_{\beta+\beta'}$ defined by (3.15), the subalgebra $\epsilon(F_{\beta+\beta'})_+$ is abelian so everything there still works.