Chapter 4

Affine W-algebras associated to truncated current Lie algebras

In this chapter, we define classical and quantum affine W-algebras associated to truncated current Lie algebras.

4.1 Vertex algebras and Poisson vertex algebras

For a vector space V, the vector space of formal Laurant series with coefficients in V is defined to be

$$V[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V \right\}.$$

It contains the following subspaces

$$V[z] = \left\{ \sum_{n=0}^{N} v_n z^n \mid v_n \in V, N \in \mathbb{Z}_{\geq 0} \right\}, \qquad V[[z]] = \left\{ \sum_{n\geq 0} v_n z^n \mid v_n \in V \right\}$$

and

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V, v_n = 0 \text{ for } n \ll 0 \right\}.$$

When V is a vector superspace, an element $v(z) = \sum_n v_n z^n \in V[[z,z^{-1}]]$ is called homogeneous if all of the coefficients v_n have the same parity, which is also defined to be the parity of $v(z)$. The formal differential and the formal residue of $v(z)$ are defined respectively as follows,

$$\partial_z v(z) := \sum_{n\in\mathbb{Z}} n v_n z^{n-1}, \qquad \mathrm{Res}_z v(z) := v_{-1}.$$

Definition 4.1.1. A *vertex superalgebra* is a quadruple $(V, |0\rangle, Y, T)$, where V is a vector superspace, $|0\rangle \in V$ is an even element called the *vacuum vector*, $Y : V \to \mathrm{End}\, V[[z,z^{-1}]]$ is a parity-preserving map sending a to $Y(a,z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ called the *vertex operator* associated to a, and $T : V \to$

V is the map defined by $Ta = a_{-2}|0\rangle$, called the *infinitesimal translation operator*. These data are required to satisfy the following axioms for all $a, b \in V$,

(i) Truncation: $a_n b = 0$ for $n \gg \infty$,

(ii) Vacuum: $T|0\rangle = 0, Y(a,z)|0\rangle|_{z=0} = a$, i.e., $a_n|0\rangle = \delta_{n,-1}a$ for $n \geq -1$,

(iii) Translation covariance: $[T, Y(a,z)] = \partial_z Y(a,z)$, i.e., $[T, a_n] = -na_{n-1}$,

(iv) Locality: $(z-w)^{N(a,b)}[Y(a,z), Y(b,w)] = 0$ for some $N(a,b) \in \mathbb{Z}_{\geq 0}$.

We call V a *vertex algebra* when V is a purely even vector space.

Here we follow the definition of V. Kac [Kac98], while R. Borcherds [Bor86] originally used the *Jacobi identity* instead of the axiom of locality: for $\ell, m, n \in \mathbb{Z}$ and $u, v \in V$,

$$\sum_{i\geq 0}(-1)^i\binom{\ell}{i}\left(u_{m+\ell-i}v_{n+i} - (-1)^{\ell+p(u)p(v)}v_{n+\ell-i}u_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}(u_{\ell+i}v)_{m+n-i}. \quad (4.1)$$

The equivalence between the Jacobi identity and the axiom of locality can be found in [DSK06]. From the Jacobi identity (4.1), one can get the following useful formulas [LL04, Kac17],

$$\text{commutator formula:} \quad [u_m, v_n] = \sum_{i\geq 0}\binom{m}{i}(u_i v)_{m+n-i}, \quad (4.2)$$

$$\text{skew-symmetry:} \quad u_m v = (-1)^{p(u)p(v)}\sum_{i=0}^{m}(-1)^{m+i+1}\frac{T^i}{i!}v_{m+i}u, \quad (4.3)$$

$$\text{iterate formula:} \quad (u_m v)_n = \sum_{i\geq 0}(-1)^i\binom{m}{i}\left(u_{m-i}v_{n+i} - (-1)^{m+p(u)p(v)}v_{m+n-i}u_i\right). \quad (4.4)$$

A vertex superalgebra V is called commutative if $[a_m, b_n] = 0$ for all $a, b \in V$ and $m, n \in \mathbb{Z}$. It is known that V is commutative if and only if $a_n = 0$ for all $a \in V$ and $n \geq 0$. Moreover, if V is not commutative, then the number $N(a,b)$ in the axiom of locality is not bounded. Indeed, a commutative vertex superalgebra is the same thing as a unital commutative associative superalgebra with a derivation. See [FHL93, LL04, Kac98] for details.

Let V be vertex superalgebra. The λ-*bracket* of $a, b \in V$ is defined to be

$$[a_\lambda b] = \sum_{n\geq 0}\frac{\lambda^n}{n!}a_n b = \mathrm{Res}_z e^{\lambda z}Y(a,z)b.$$

By the truncation axiom, $[a_\lambda b] \in V[\lambda]$ is a polynomial in λ with coefficients in V. The λ-bracket satisfies the following properties [Kac17], for all $a, b, c \in V$,

$$\text{sesquilinearity:} \quad [(Ta)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda(Tb)] = (\lambda + T)[a_\lambda b], \quad (4.5)$$

$$\text{skew-symmetry:} \quad [b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-T}b], \quad (4.6)$$

$$\text{Jacobi identity:} \quad [a_\lambda[b_\mu c]] - (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu}c]. \quad (4.7)$$

Definition 4.1.2. A *Lie conformal superalgebra* is a vector superspace R admitting a $\mathbb{C}[T]$-module structure, where T is an indeterminate that acts on R as an even endomorphism, endowed with a $\mathbb{C}$-bilinear λ-bracket $[\cdot_\lambda\cdot] : R \otimes R \to \mathbb{C}[\lambda] \otimes R$, such that (4.5), (4.6) and (4.7) are satisfied.

Let V be a vertex superalgebra and $a(z) = Y(a,z), b(z) = Y(b,z)$ for $a, b \in V$. The *normal ordered product* of $a(z)$ and $b(z)$ is defined to be

$$: a(z)b(z) := a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z)a(z)_-,$$

where $a(z)_+ := \sum_{n<0} a_n z^{-n-1}$ and $a(z)_- := \sum_{n\geq 0} a_n z^{-n-1}$. One can show that $: a(z)b(z) := Y(a_{-1}b, z)$. The normal ordered product of several vertex operators is defined from right to left, and we have $: a(z)b(z)c(z) := Y(a_{-1}(b_{-1}c), z)$.

The coefficients of $Y(a,z)$ are called the *Fourier coefficients* or *modes* of a. The zero mode a_0 will play an important role in the sequel. The minus one mode a_{-1} is usually considered as a product in V. Indeed, $: ab := a_{-1}b$ defines an algebra structure on V, where the vacuum vector $|0\rangle$ plays the role of unit and T plays a role of derivation, i.e., $(V, |0\rangle, ::, T)$ is a unital differential superalgebra. With respect to $::$, V is usually neither commutative nor associative, but we have (see [Kac17])

$$\text{weak-commutativity:} \quad : ab : -(-1)^{p(a)p(b)} : ba := \sum_{n\geq 0} (-1)^n \frac{T^{n+1}}{(n+1)!} a_n b = \int_{-T}^{0} [a_\lambda b] d\lambda, \tag{4.8}$$

$$\begin{aligned} \text{weak-associativity:} \quad & :: ab : c : - : a : bc :: \\ &= \sum_{n\geq 0} \left(: \frac{T^{n+1}a}{(n+1)!} (b_n c) : + (-1)^{p(a)p(b)} : \frac{T^{n+1}b}{(n+1)!} (a_n c) : \right) \\ &=: \left(\int_0^T d\lambda a \right) [b_\lambda c] : + (-1)^{p(a)p(b)} : \left(\int_0^T d\lambda b \right) [a_\lambda c] : . \end{aligned} \tag{4.9}$$

Remark 4.1.3. *Given a polynomial $f(\lambda) = \sum_{i=0}^{n} \lambda^i v_i \in V[\lambda]$, where V is a vector space, we define*

$$\int_A^B f(\lambda) d\lambda = \sum_{i=0}^{n} \frac{B^{i+1} - A^{i+1}}{i+1} v_i.$$

When A, B are operators acting on V, we write $d\lambda$ before the element they act on if it is not clear. For example, in (4.9), T acts on a in the first term and on b in the second term.

The following is a λ-bracket version of the definition of vertex superalgebras [DSK06].

Definition 4.1.4. A vertex superalgebra is a quintuple $(V, |0\rangle, T, [\cdot_\lambda\cdot], ::)$, such that

(i) $(V, [\cdot_\lambda\cdot], T)$ is a Lie conformal superalgebra by considering V as a $\mathbb{C}[T]$-module,

(ii) $(V, |0\rangle, ::, T)$ is a unital differential superalgebra satisfying (4.8) and (4.9),

(iii) The product :: and the λ-bracket $[\cdot_\lambda\cdot]$ are related by the non-commutative Wick formula

$$[a_\lambda :bc:] =: [a_\lambda b]c: + (-1)^{p(a)p(b)} :b[a_\lambda c]: + \int_0^\lambda [[a_\lambda b]_\mu c]d\mu. \tag{4.10}$$

A Poisson algebra (Definition 1.1.1) is a commutative associative algebra with another Lie bracket such that the associative multiplication and the Lie bracket satisfy Leibniz's rule. The notion of a Poisson vertex superalgebra can be introduced in a similar way.

Definition 4.1.5 ([Li05])**.** A *vertex Lie superalgebra* is a triple (V, Y_-, D), where V is a vector superspace, D is a linear operator $D : V \to V$ and Y_- is a parity-preserving linear map

$$Y_- : V \to \operatorname{End} V[[z, z^{-1}]], \qquad v \mapsto Y_-(v, z) = \sum_{n\geq 0} v_{[n]} z^{-n-1},$$

such that $u_{[n]}v = 0$ for $n \gg 0$, $(Dv)_{[n]} = -nv_{[n-1]}$ for $u, v \in V$ and $n \geq 0$. Moreover, we require (4.2) and (4.3) to be satisfied for all $u, v \in V$ and $m, n \in \mathbb{Z}_{\geq 0}$ if we replace a_n by $a_{[n]}$, etc.

Remark 4.1.6. *Note that we use a_n for the Fourier coefficients of the linear map Y in a vertex superalgebra, and $a_{[n]}$ for the coefficients of the linear map Y_- in a vertex Lie superalgebra.*

Definition 4.1.7. A *Poisson vertex superalgebra* is a commutative vertex superalgebra $(V, |0\rangle, Y, T)$, with a vertex Lie superalgebra structure (V, Y_-, T) such that for all $a, b, c \in V$ and $n \geq 0$, we have

$$a_{[n]}(b_{-1}c) = (a_{[n]}b)_{-1}c + (-1)^{p(a)p(b)} b_{-1}(a_{[n]}c).$$

Notation: For a Poisson vertex superalgebra $(V, |0\rangle, Y, Y_-, T)$, since $a_n = 0$ for all $a \in V$ and $n \geq 0$, where a_n is the Fourier coefficients of the vertex operator $Y(a, z)$, we denote by $a_n = a_{[n]}$ for $n \geq 0$, where $a_{[n]}$ is the Fourier coefficients of $Y_-(a, z)$.

Note that in a Poisson vertex superalgebra V, (4.8) and (4.9) become

$$:ab: = (-1)^{p(a)p(b)} :ba: \quad \text{and} \quad ::ab:c: =: a:bc::,$$

so $(V, ::)$ is both commutative and associative. For $a, b \in V$, we denote by

$$\{a_\lambda b\} = \sum_{n\geq 0} a_n b. \tag{4.11}$$

One can show that $\{\cdot_\lambda\cdot\}$ also satisfies (4.5), (4.6) and (4.7), and we have the following equivalent definition of a Poisson vertex superalgebra [DSK06].

Definition 4.1.8. A Poisson vertex superalgebra is a quintuple $(V, |0\rangle, T, \{\cdot_\lambda\cdot\}, ::)$ such that

(i) $(V, |0\rangle, ::, T)$ is a unital associative and commutative differential superalgebra,

(ii) $(V, \{\cdot_\lambda\cdot\}, T)$ is a Lie conformal superalgebra,

(iii) The product :: and the λ-bracket $\{\cdot_\lambda\cdot\}$ are related by the commutative Wick formula

$$\{a_\lambda : bc :\} =: \{a_\lambda b\}c : + (-1)^{p(a)p(b)} : b\{a_\lambda c\} : . \tag{4.12}$$

Remark 4.1.9. *Compared to the λ-bracket definition of a vertex superalgebra, we do not have the integral terms in (4.8), (4.9) and (4.10) for a Poisson vertex superalgebra.*

Let V be a vertex superalgebra. The commutator formula (4.2) implies that

$$[a_0, b_n] = (a_0 b)_n. \tag{4.13}$$

Lemma 4.1.10. *Let $(V, |0\rangle, Y, T)$ be a vertex superalgebra and $d \in V$ satisfying $d_0^2 = 0$. Then the homology $H(V, d_0) := \dfrac{\ker\ d_0}{\operatorname{im} d_0}$ inherits a vertex superalgebra structure from that of V.*

Proof. Recall that $Y(d, z) = \sum_{n\in\mathbb{Z}} d_n z^{-n-1}$ and d_0 is the zero mode of d. It is enough to prove that for all $a \in \ker\ d_0$, $Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ is a well-defined element in $\operatorname{End} H(V, d_0)[[z, z^{-1}]]$, i.e., a_n preserves both $\ker\ d_0$ and $\operatorname{im} d_0$. From (4.13), for all $b \in V$, we have

$$[d_0, a_n]b = (d_0 a)_n b = 0, \quad \text{i.e.,} \quad d_0(a_n b) = (-1)^{p(d)p(a)} a_n(d_0 b). \tag{4.14}$$

Let $u \in \ker\ d_0$ and $v \in \operatorname{im} d_0$ with $v = d_0 w$. Then (4.14) implies that $a_n u \in \ker\ d_0$ and

$$a_n v = a_n(d_0 w) = (-1)^{p(d)p(a)} d_0(a_n w) \in \operatorname{im} d_0.$$

Therefore, $Y(a, z)$ is a well-defined element in $\operatorname{End} H(V, d_0)[[z, z^{-1}]]$, for all $a \in \ker\ d_0$. □

Remark 4.1.11. *$H(V, d_0)$ is a Poisson vertex superalgebra if V is a Poisson vertex superalgebra.*

4.2 Non-linear Lie conformal algebras and their universal enveloping vertex algebras

Definition 4.2.1. A *non-linear Lie superalgebra* is a vector superspace L, equipped with a parity-preserving linear map $[\cdot,\cdot]$, $L \otimes L \to L \oplus \mathbb{C}$, where $\mathbb{C}$ is defined to be even, such that for all $a, b, c \in L$, we have $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ and the Jacobi identity holds:

$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]. \tag{4.15}$$

We assume that $[\mathbb{C}, L] = 0$ in the Jacobi identity. Note that $L \oplus \mathbb{C}$ is a Lie superalgebra.

Definition 4.2.2. A *non-linear Lie conformal superalgebra* is a $\mathbb{C}[T]$-module R, endowed with a λ-bracket $[\cdot_\lambda\cdot] : R \otimes R \to \mathbb{C}[\lambda] \otimes (R \oplus \mathbb{C})$, such that (4.5), (4.6) and (4.7) are satisfied.

Remark 4.2.3. *In the definition of a non-linear Lie conformal superalgebra, we can understand that $T\mathbb{C} = 0$ so that $R \oplus \mathbb{C}$ is a Lie conformal superalgebra. We should also understand that $[\mathbb{C}_\lambda R] = 0$ in the Jacobi identity. There is a more general version of a non-linear Lie conformal algebra in [DSK06].*

Given a non-linear Lie conformal superalgebra R, define a bracket $[\cdot,\cdot] : R \otimes R \to R \oplus \mathbb{C}$ by

$$[a,b] := -\sum_{n\geq 0} \frac{(-T)^{n+1}}{(n+1)!} a_n b = \int_{-T}^{0} [a_\lambda b] d\lambda. \tag{4.16}$$

Lemma 4.2.4 ([BK03]). *The Lie bracket (4.16) defines a non-linear Lie superalgebra structure on R.*

Proof. The bracket is obviously bilinear and takes values in $R \oplus \mathbb{C}$. We only need to verify skew-symmetry and the Jacobi identity. We have

$$\int_{-T}^{0} [a_\lambda b] d\lambda = -(-1)^{p(a)p(b)} \int_{-T}^{0} [b_{-\lambda-T} a] d\lambda = (-1)^{p(a)p(b)} \int_{0}^{-T} [b_\mu a] d\mu,$$

i.e., $[a,b] = -(-1)^{p(a)p(b)}[b,a]$.

For the Jacobi identity, note that

$$[a,[b,c]] = \int_{-T}^{0} \left[a_\lambda \int_{-T}^{0} [b_\mu c] d\mu \right] d\lambda = \int_{-T}^{0} \int_{-\lambda-T}^{0} [a_\lambda [b_\mu c]] d\mu d\lambda.$$

Similarly, we have

$$\begin{aligned} [[a,b],c] &= \int_{-T}^{0} \left[\int_{-T}^{0} [a_\lambda b] d\lambda_\mu c \right] d\mu \\ &= \int_{-T}^{0} \int_{\mu}^{0} [[a_\lambda b]_\mu c]] d\lambda d\mu \\ &= \int_{-T}^{0} \int_{-T}^{\lambda} [[a_\lambda b]_\mu c]] d\mu d\lambda \\ &= \int_{-T}^{0} \int_{-\lambda-T}^{0} [[a_\lambda b]_{\mu'+\lambda} c]] d\mu' d\lambda, \end{aligned}$$

and

$$\begin{aligned} [b,[a,c]] &= (-1)^{p(a)p(b)} \int_{-T}^{0} \left[b_\mu \int_{-T}^{0} [a_\lambda c] d\lambda \right] d\mu \\ &= (-1)^{p(a)p(b)} \int_{-T}^{0} \int_{-\mu-T}^{0} [b_\mu [a_\lambda c]] d\lambda d\mu \\ &= (-1)^{p(a)p(b)} \int_{-T}^{0} \int_{-\lambda-T}^{0} [b_\mu [a_\lambda c]] d\mu d\lambda. \end{aligned}$$

Now the Jacobi identity (4.15) comes from the Jacobi identity (4.7). □

The non-linear Lie superalgebra structure on R defined by (4.16) is denoted by R_{Lie}.

Remark 4.2.5. *Let $(R, [\cdot_\lambda\cdot], T)$ be a non-linear Lie conformal superalgebra. Another construction of R_{Lie} is as follows [Kac98]. Let $\tilde{R} = (R \oplus \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$, where t and t^{-1} are considered as even elements. Let $\tilde{T} = T \otimes 1 + 1 \otimes \partial_t$. Define*

$$(a \otimes f)_n (b \otimes g) = \sum_{i\geq 0} (a_{n+i} b) \otimes \left(\frac{\partial_t^i f}{i!} g \right)$$

and

$$[(a\otimes f)_\lambda(b\otimes g)] = \sum_{n\geq 0}\frac{\lambda^n}{n!}(a\otimes f)_n(b\otimes g).$$

Then one can show [Kac98] that $\tilde{R}$ is a Lie conformal superalgebra and $Lie\, R = \tilde{R}/\tilde{T}\tilde{R}$ is a Lie superalgebra with Lie bracket

$$[a\otimes t^m, b\otimes t^n] = \sum_{i\geq 0}(a_i b)\otimes t^{m+n-i}.$$

Assume that $R = (\mathbb{C}[T]\otimes U)\oplus S$, where $\mathbb{C}[T]\otimes U$ is the free part and S the torsion part of R as a $\mathbb{C}[T]$-module. Then it can be proved that

$$Lie\, R \cong U\otimes[t,t^{-1}]\oplus S\otimes t^{-1}\oplus\mathbb{C}\otimes t^{-1} \tag{4.17}$$

as vector spaces. Let $(Lie\, R)_-$ be the $\mathbb{C}$-span of the images of $a\otimes t^n$ for $a\in R$ and $n\geq 0$, and $(Lie\, R)_+$ be the $\mathbb{C}$-span of the images of $a\otimes t^n$ for $a\in R$ and $n<0$. Then both $(Lie\, R)_\pm$ are non-linear Lie subalgebras of $Lie\, R$ (by identifying $\mathbb{C}t^{-1}$ with $\mathbb{C}$). The subalgebra $(Lie\, R)_-$ is called the annihilation algebra of R and it plays important roles in the representation theory of R. The subalgebra $(Lie\, R)_+$ is isomorphic to R_{Lie} by sending $s\otimes t^{-1}$ to s for $s\in S$ and $u\otimes t^{-n}$ to $\dfrac{(-T)^{n-1}u}{(n-1)!}$ for $u\in U$ and $n\geq 1$.

Let R be a non-linear Lie conformal superalgebra, and $S(R)$ the symmetric algebra of R. Then $S(R)$ is naturally a commutative superalgebra. The action of T on R can be extended to $S(R)$ by requiring $T(ab) = T(a)b + aT(b)$ for all $a, b\in S(R)$. Therefore, $S(R)$ is a unital commutative differential associative superalgebra hence a commutative vertex superalgebra. Define a λ-bracket on $S(R)$ by letting $\{\cdot_\lambda\cdot\} = [\cdot_\lambda\cdot] : R\times R\to S(R)$ be the λ-bracket of R, and then extend it to $S(R)\times S(R)$ by requiring (4.12). Then one can show that these data define a Poisson vertex superalgebra on $S(R)$.

Proposition 4.2.6. *Let R be a non-linear Lie conformal superalgebra, and $S(R)$ the symmetric algebra of R. Then there is a Poisson vertex superalgebra structure on $S(R)$, such that $\{\cdot_\lambda\cdot\} : R\times R\to S(R)$ is the λ-bracket of R.*

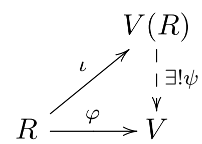
Let R_{Lie} be the non-linear Lie superalgebra defined by (4.16). The universal enveloping algebra of R_{Lie} is defined to be $U(R_{Lie}) = T(R_{Lie})/I$, where $T(R_{Lie})$ is the tensor algebra of R_{Lie} and I is the two-sided ideal of $T(R_{Lie})$ generated by $a\otimes b - (-1)^{p(a)p(b)}b\otimes a - [a,b]$ for all $a, b\in R$.

Proposition 4.2.7 ([BK03])**.** *Let R be a non-linear Lie conformal superalgebra. Then the universal enveloping algebra $U(R_{Lie})$ of R_{Lie} has a vertex superalgebra structure, where the λ-bracket on $R_{Lie}\times R_{Lie}$ is the λ-bracket of R, and the product :: on $R_{Lie}\times U(R_{Lie})$ is the product of $U(R_{Lie})$.*

Proof. The unit element of $U(R_{Lie})$ plays the role of the vacuum vector. The λ-bracket and the product :: can be extended to $U(R_{Lie})$ by (4.9) and (4.10) in a unique way. □

Definition 4.2.8. The vertex superalgebra $U(R_{Lie})$ is called the *universal enveloping vertex superalgebra* of R, and is usually denoted by $V(R)$.

Remark 4.2.9. *Like the universal enveloping algebra of a Lie algebra, $V(R)$ has the following universal property: (1) The natural inclusion $\iota : R \oplus \mathbb{C} \hookrightarrow V(R)$, while $\mathbb{C} \to \mathbb{C}|0\rangle$, is a Lie conformal superalgebra homomorphism; (2) Let V be a vertex superalgebra and $\varphi : R \oplus \mathbb{C} \to V$ a Lie conformal superalgebra homomorphism with $\mathbb{C} \to \mathbb{C}|0\rangle$. Then there exists a unique vertex superalgebra homomophism $\psi : V(R) \to V$ such that $\psi \circ \iota = \varphi$.*



Let L be a non-linear Lie superalgebra with an invariant supersymmetric bilinear form $(\cdot \mid \cdot)$. Let $k \in \mathbb{C}$ and $\operatorname{Cur}^k L := \mathbb{C}[T] \otimes L$. Set

$$[a_\lambda b] := [a, b] + \lambda k(a \mid b) \quad \text{ for } a, b \in L. \tag{4.18}$$

Lemma 4.2.10. *There is a unique non-linear Lie conformal superalgebra structure on $\operatorname{Cur}^k L$ satisfying (4.18).*

Proof. Once the λ-bracket is well-defined for all $a, b \in L$, it extends uniquely to a λ-bracket on $\operatorname{Cur}^k L$ by (4.5). Skew-symmetry of $[\cdot_\lambda \cdot]$ comes from the skew-symmetry of the Lie bracket $[\cdot, \cdot]$ and the supersymmetry of $(\cdot \mid \cdot)$. The Jacobi identity of $[\cdot_\lambda \cdot]$ comes from the Jacobi identity of $[\cdot, \cdot]$. □

Let A be a finite-dimensional vector superspace and $\langle \cdot, \cdot \rangle$ a non-degenerate skew-supersymmetric bilinear form on A. Let $R(A) := \mathbb{C}[T] \otimes A$ and define

$$[a_\lambda b] := \langle a, b \rangle \quad \text{ for } a, b \in A. \tag{4.19}$$

Lemma 4.2.11. *There is a unique non-linear Lie conformal superalgebra structure on $R(A)$ satisfying (4.19).*

Proof. Once the λ-bracket for all $a, b \in A$ are well-defined, it extends uniquely to a λ-bracket on $R(A)$ by (4.5). Skew-symmetry of $[\cdot_\lambda \cdot]$ comes from the skew-symmetry of $\langle \cdot, \cdot \rangle$. The Jacobi identity is trivial since we assume that $[\mathbb{C}_\lambda R] = 0$ in all non-linear Lie conformal superalgebras. □

Now let us consider the universal enveloping vertex superalgebras of $\operatorname{Cur}^k L$ and $R(A)$.

Example 4.2.12. Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$. The *Kac-Moody affinization* of $\mathfrak{g}$ is the Lie algebra

$$\hat{\mathfrak{g}} = \left(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\right) \oplus \mathbb{C}K$$

with Lie bracket:

$$[at^m, bt^n] = [a,b]t^{m+n} + m\delta_{m,-n}(a \mid b)K, \quad [K, \hat{\mathfrak{g}}] = 0.$$

Let $\hat{\mathfrak{g}}_+ = (\mathfrak{g} \otimes \mathbb{C}[t]) \oplus \mathbb{C}K$, which is a subalgebra of $\hat{\mathfrak{g}}$. Define a one-dimensional module $\mathbb{C}_k$ of $\hat{\mathfrak{g}}_+$ on which $\mathfrak{g} \otimes \mathbb{C}[t]$ acts as zero and K acts as the constant k. The *level k vacuum representation* of $\tilde{\mathfrak{g}}$ is the induced module

$$V^k(\mathfrak{g}) := \mathrm{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbb{C}_k.$$

It is isomorphic to $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ as a vector space. There is a unique vertex algebra structure on $V^k(\mathfrak{g})$ with the vacuum vector being the identity 1 and $a(z) := Y(at^{-1}, z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$ for all $a \in \mathfrak{g}$. It is called the *universal affine vertex algebra* of level k associated to $\mathfrak{g}$.

Remark 4.2.13. *Note that $Lie\,\mathrm{Cur}^k\,\mathfrak{g} \cong \hat{\mathfrak{g}}$ by considering $k \otimes t^{-1}$ as K, where $k \otimes t^{-1}$ is defined by (4.17). We have $(Lie\,\mathrm{Cur}^k\,\mathfrak{g})_- \cong \hat{\mathfrak{g}}_+$ and $(Lie\,\mathrm{Cur}^k\,\mathfrak{g})_+ \cong \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$. Since $(\mathrm{Cur}^k\,\mathfrak{g})_{Lie} \cong (Lie\,\mathrm{Cur}^k\,\mathfrak{g})_+$, we have $V^k(\mathfrak{g}) \cong U((\mathrm{Cur}^k\,\mathfrak{g})_{Lie})$ as vector spaces. Comparing λ-brackets of $V^k(\mathfrak{g})$ and $U((\mathrm{Cur}^k\,\mathfrak{g})_{Lie})$, one can see that they are isomorphic as vertex algebras.*

Example 4.2.14. Let A be a finite-dimensional vector superspace and $\langle\cdot,\cdot\rangle$ be a non-degenerate skew-supersymmetric bilinear form on A. The *Clifford affinization* of A is the Lie superalgebra $\widehat{A} := (A \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K$ with Lie bracket:

$$[at^m, bt^n] = \langle a, b\rangle \delta_{m,-n-1}K, \qquad [K, \widehat{A}] = 0.$$

Let $\widehat{A}_+ = (A \otimes \mathbb{C}[t]) \oplus \mathbb{C}K$, which is an abelian subalgebra of $\widehat{A}$. Let $\mathbb{C}$ be the one-dimensional representation of $\widehat{A}_+$, on which $A \otimes \mathbb{C}[t]$ acts as zero and K acts as the identity. The *Fock representation* of $\widehat{A}$ is the induced module

$$F(A) := \mathrm{Ind}_{\widehat{A}_+}^{\widehat{A}} \mathbb{C}.$$

It is isomorphic to $U(A \otimes t^{-1}\mathbb{C}[t^{-1}])$ as a vector space. There is a unique vertex superalgebra structure on $F(A)$ with the vacuum vector being the identity 1 and $a(z) := Y(at^{-1}, z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$ for $a \in A$. It is called the *vertex superalgebra of fermions* associated to A and $\langle\cdot,\cdot\rangle$.

Remark 4.2.15. *As in Remark 4.2.13, we have $F(A) \cong U(R(A)_{Lie})$. Indeed, we have $Lie\,R(A) \cong \widehat{A}$ by considering $1 \otimes t^{-1}$ as K, where $1 \otimes t^{-1}$ is defined in (4.17). Moreover, we have $(Lie\,R(A))_- \cong \widehat{A}_+$ and $(Lie\,R(A))_+ \cong A \otimes t^{-1}\mathbb{C}[t^{-1}]$. Since $R(A)_{Lie} \cong (Lie\,R(A))_+$, we have $F(A) \cong U(R(A)_{Lie})$ as vector spaces. Comparing their λ-brackets shows that they are isomorphic as vertex superalgebras.*

Here are two examples.

Example 4.2.16. Let A^{ne} be a finite-dimensional vector space and $\langle\cdot,\cdot\rangle$ be a non-degenerate skew-symmetric bilinear form on A^{ne}. By Lemma 4.2.11, we have a non-linear Lie conformal algebra $R(A^{ne})$. By Example 4.2.14, we have its universal enveloping vertex algebra $F(A^{ne})$, which is called the *vertex algebra of neutral fermions* associated to A^{ne} and $\langle\cdot,\cdot\rangle$.

Example 4.2.17. Let V be a finite-dimensional vector space and V^* be its dual. Let $\iota(V)$ and $\varepsilon(V^*)$ be copies of V and V^*, respectively, but considered as purely odd spaces. For $v \in V$ and $u^* \in V^*$, let $\iota(v)$ and $\varepsilon(u^*)$ be the corresponding elements in $\iota(V)$ and $\varepsilon(V^*)$. Let $A^{ch} = \iota(V) \oplus \varepsilon(V^*)$, and endow A^{ch} with a non-degenerate skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ by setting $\langle \iota(V), \iota(V) \rangle = \langle \varepsilon(V^*), \varepsilon(V^*) \rangle = 0$ and $\langle \iota(v), \varepsilon(u^*) \rangle = \langle \varepsilon(u^*), \iota(v) \rangle = u^*(v)$ for all $v \in V, u^* \in V^*$. By Lemma 4.2.11, we have a non-linear Lie conformal superalgebra $R(A^{ch})$. By Example 4.2.14, we have its universal enveloping vertex superalgebra $F(A^{ch})$, which is called the *vertex superalgebra of charged superfermions* associated to A^{ch}.

4.3 Affine W-algebras associated to truncated current Lie algebras

Now let us come back to the basic setting of Chapter 2. Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$ with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$. Let $\mathfrak{g}_p$ be the level p truncated current Lie algebra associated to $\mathfrak{g}$ and $(\cdot \mid \cdot)_p$ a fixed non-degenerate invariant bilinear form on $\mathfrak{g}_p$. Let $\Gamma : \mathfrak{g} \overset{\operatorname{ad} h_\Gamma}{=\!=} \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be a good $\mathbb{Z}$-grading of $\mathfrak{g}$ with a good element $e \in \mathfrak{g}(2)$, and $\{e, f, h\}$ an $s\ell_2$-triple with $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-2)$. Let

$$\mathfrak{g}_p = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_p(i), \text{ where } \mathfrak{g}_p(i) := \{y \in \mathfrak{g}_p \mid [h_\Gamma, y] = iy\} = \mathfrak{g}(i)_p \tag{4.20}$$

be the corresponding good $\mathbb{Z}$-grading of $\mathfrak{g}_p$, with the same good element e. By Lemma 2.2.5, the bilinear form $\langle \cdot, \cdot \rangle_p$ on $\mathfrak{g}_p(-1)$ defined by $\langle a, b \rangle_p = (e \mid [a, b])_p$ for $a, b \in \mathfrak{g}_p(-1)$ is non-degenerate.

Let $A_p^{ne} = \epsilon(\mathfrak{g}_p(-1))$ be a copy of $\mathfrak{g}_p(-1)$. For $x \in \mathfrak{g}_p(-1)$, let $\epsilon(x)$ be the corresponding element of A_p^{ne}. More generally, for $x \in \mathfrak{g}_p$, we write $\epsilon(x) = \epsilon(x_{-1})$, where $x = \sum_i x_i$ and $x_i \in \mathfrak{g}_p(i)$. The form $\langle \epsilon(x), \epsilon(y) \rangle_p := \langle x, y \rangle_p$ is skew-symmetric and non-degenerate on A_p^{ne}. By Lemma 4.2.11 and Example 4.2.16, we have a non-linear Lie conformal algebra $R(A_p^{ne})$ and its universal enveloping vertex algebra $F(A_p^{ne})$.

Let $\mathfrak{n}_p = \bigoplus_{j<0} \mathfrak{g}_p(j)$ and $\mathfrak{n}_p^*$ be the dual of $\mathfrak{n}_p$. Let $\iota(\mathfrak{n}_p)$ and $\varepsilon(\mathfrak{n}_p^*)$ be copies of $\mathfrak{n}_p$ and $\mathfrak{n}_p^*$, respectively, but considered as purely odd spaces. For $u \in \mathfrak{n}_p$ and $v^* \in \mathfrak{n}_p^*$, let $\iota(u)$ and $\varepsilon(v^*)$ be the corresponding elements of $\iota(\mathfrak{n}_p)$ and $\varepsilon(\mathfrak{n}_p^*)$, respectively. Let $A_p^{ch} = \iota(\mathfrak{n}_p) \oplus \varepsilon(\mathfrak{n}_p^*)$. By Lemma 4.2.11 and Example 4.2.17 we have a non-linear Lie conformal superalgebra $R(A_p^{ch})$ and its universal enveloping vertex superalgebra $F(A_p^{ch})$.

For $\mathfrak{g}_p$ and the bilinear form $(\cdot \mid \cdot)_p$, we have the non-linear Lie conformal algebra $\mathrm{Cur}^k \mathfrak{g}_p$ by Lemma 4.2.10 and its universal enveloping vertex algebra $V^k(\mathfrak{g}_p)$ by Example 4.2.12.

Let us choose a basis $\{u_\alpha\}_{\alpha \in S_p^j}$ of $\mathfrak{g}_p(j)$ for each j. Let $S_p^- = \bigcup_{j<0} S_p^j$ and $S_p' = S_p^{-1}$. Then $\{u_\alpha\}_{\alpha \in S_p^-}$ forms a basis of $\mathfrak{n}_p$ and $\{\epsilon(u_\alpha)\}_{\alpha \in S_p'}$ forms a basis of A_p^{ne}. Let $\{u_\alpha^*\}_{\alpha \in S_p^-}$ be the dual basis of $\mathfrak{n}_p^*$, with $\langle u_\alpha^*, u_\beta \rangle = \delta_{\alpha,\beta}$. Then $\{\iota(u_\alpha)\}_{\alpha \in S_p^-}$ and $\{\varepsilon(u_\alpha^*)\}_{\alpha \in S_p^-}$ form dual bases of $\iota(\mathfrak{n}_p)$ and $\varepsilon(\mathfrak{n}_p^*)$, respectively. Let $\{c_{i,j}^k\}$ be the structure constants of $\mathfrak{g}_p$ with respect to the basis $\{u_i\}$, i.e., $[u_i, u_j] = \sum_k c_{i,j}^k u_k$.

4.3.1 Classical affine W-algebras through classical Drinfeld-Sokolov reduction

Let

$$R^k(\mathfrak{g}_p, e) = \mathrm{Cur}^k\, \mathfrak{g}_p \oplus R(A_p^{ch}) \oplus R(A_p^{ne}) = \mathbb{C}[T] \otimes \Big(\mathfrak{g}_p \oplus A_p^{ch} \oplus A_p^{ne}\Big)$$

be the direct sum of three non-linear Lie conformal superalgebras and $S(R^k(\mathfrak{g}_p, e))$ its symmetric algebra. By Proposition 4.2.6, $S(R^k(\mathfrak{g}_p, e))$ has a Poisson vertex superalgebra structure.

Let

$$\bar{d}^p = \sum_{i \in S_p^-} \varepsilon(u_i^*)\,(u_i + (e \mid u_i)_p + \epsilon(u_i)) - \frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j])\varepsilon(u_i^*)\varepsilon(u_j^*).$$

Obviously, $\bar{d}^p \in S(R^k(\mathfrak{g}_p, e))$ is an odd element.

Lemma 4.3.1. *We have the following formulas for the λ-bracket of $\bar{d}^p$ in $S(R^k(\mathfrak{g}_p, e))$,*

(1) $\{\bar{d}^p{}_\lambda x\} = \sum_{i \in S_p^-} ([u_i, x] + k(x \mid u_i)_p(\lambda + T))\varepsilon(u_i^*)$ *for* $x \in \mathfrak{g}_p$*;*

(2) $\{\bar{d}^p{}_\lambda \epsilon(y)\} = \sum_{i \in S_p^-} (e \mid [u_i, y])_p \varepsilon(u_i^*)$ *for* $y \in \mathfrak{g}_p(-1)$*;*

(3) $\{\bar{d}^p{}_\lambda \varepsilon(v^*)\} = -\frac{1}{2} \sum_{i,j \in S_p^-} \langle v^*, [u_i, u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*)$ *for* $v^* \in \mathfrak{n}_p^*$*;*

(4) $\{\bar{d}^p{}_\lambda \iota(u)\} = \sum_{i,j \in S_p^-} \langle u_i^*, u\rangle \Big(\iota([u_i, u_j])\varepsilon(u_j^*) + u_i + (e \mid u_i)_p + \epsilon(u_i)\Big)$ *for* $u \in \mathfrak{n}_p$*.*

Proof. Let $X(u_i) = u_i + (e \mid u_i)_p + \epsilon(u_i)$ and write $\bar{d}^p = \bar{d}^{p,1} + \bar{d}^{p,2}$, where

$$\bar{d}^{p,1} = \sum_{i \in S_p^-} \varepsilon(u_i^*) X(u_i) \quad \text{and} \quad \bar{d}^{p,2} = -\frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j])\varepsilon(u_i^*)\varepsilon(u_j^*).$$

Instead of calculating $\{\bar{d}^p{}_\lambda \cdot\}$, we will calculate $\{\cdot_\lambda \bar{d}^p\}$ and then use skew-symmetry to get the formulas for $\{\bar{d}^p{}_\lambda \cdot\}$. Our calculations are based on (4.12).

We have $\{a_\lambda b\} = 0$ if a, b come from different summands of $R^k(\mathfrak{g}_p, e)$. In particular, $\{x_\lambda \bar{d}^{p,2}\} = \{\epsilon(y)_\lambda \bar{d}^{p,2}\} = 0$ for all $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_p(-1)$. By (4.12), we have

$$\{x_\lambda \bar{d}^p\} = \{x_\lambda \bar{d}^{p,1}\} = \sum_{i \in S_p^-} \varepsilon(u_i^*)\{x_\lambda X(u_i)\} = \sum_{i \in S_p^-} \varepsilon(u_i^*)([x, u_i] + \lambda k(x \mid u_i)_p) \tag{4.21}$$

and

$$\{\epsilon(y)_\lambda \bar{d}^p\} = \{\epsilon(y)_\lambda \bar{d}^{p,1}\} = \sum_{i \in S_p^-} \varepsilon(u_i^*)\{\epsilon(y)_\lambda X(u_i)\} = \sum_{i \in S_p^-} (e \mid [y, u_i])_p \varepsilon(u_i^*). \tag{4.22}$$

We obviously have $\{\varepsilon(v^*)_\lambda \bar{d}^{p,1}\} = 0$, so

$$\{\varepsilon(v^*)_\lambda \bar{d}^p\} = -\frac{1}{2} \sum_{i,j \in S_p^-} \{\varepsilon(v^*)_\lambda \iota([u_i, u_j])\}\varepsilon(u_i^*)\varepsilon(u_j^*) = -\frac{1}{2} \sum_{i,j \in S_p^-} \langle v^*, [u_i, u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*). \tag{4.23}$$

Finally, since $\{\iota(u)_\lambda X(u_i)\} = 0$, we have

$$\{\iota(u)_\lambda \bar{d}^{p,1}\} = \sum_{i\in S_p^-} \{\iota(u)_\lambda \varepsilon(u_i^*)\} X(u_i) = \sum_{i\in S_p^-} \langle u_i^*, u\rangle X(u_i), \tag{4.24}$$

and

$$\begin{aligned}\{\iota(u)_\lambda \bar{d}^{p,2}\} &= \frac{1}{2}\sum_{i,j\in S_p^-} \iota([u_i,u_j])\langle u_i^*, u\rangle \varepsilon(u_j^*) - \frac{1}{2}\sum_{i,j\in S_p^-} \iota([u_i,u_j])\varepsilon(u_i^*)\langle u_j^*, u\rangle \\ &= \sum_{i,j\in S_p^-} \langle u_i^*, u\rangle \iota([u_i,u_j])\varepsilon(u_j^*). \end{aligned} \tag{4.25}$$

Applying skew-symmetry to (4.21), (4.22), (4.23) and (4.24)+(4.25), we get the desired formulas for $\{\bar{d}^p{}_\lambda \cdot\}$ in $S(R^k(\mathfrak{g}_p, e))$. □

Before stating the following proposition, let us recall that in a (Poisson) vertex superalgebra, if v is an odd element and satisfies $\{v_\lambda v\} = 0$, then $(v_0)^2 = 0$. Indeed, $\{v_\lambda v\} = 0$ implies that $v_0 v = 0$. But $2(v_0)^2 = v_0 v_0 + v_0 v_0 = [v_0, v_0] = (v_0 v)_0 = 0$ by (4.13).

Proposition 4.3.2. *We have* $\{\bar{d}^p{}_\lambda \bar{d}^p\} = 0$ *hence* $(\bar{d}^p_0)^2 = 0$.

Proof. Recall that $X(u_i) = u_i + (e \mid u_i)_p + \epsilon(u_i)$ and $\{c_{i,j}^k\}$ are the structure constants of $\mathfrak{g}_p$ with respect to the basis $\{u_i\}$. Using (4.12) and the formulas in Lemma 4.3.1, we have

$$\begin{aligned}\{\bar{d}^p{}_\lambda \bar{d}^{p,1}\} &= \sum_{\ell\in S_p^-} \{\bar{d}^p{}_\lambda \varepsilon(u_\ell^*)\} X(u_\ell) - \sum_{i\in S_p^-} \varepsilon(u_i^*)\{\bar{d}^p{}_\lambda X(u_i)\} \\ &= -\frac{1}{2}\sum_{i,j,\ell\in S_p^-} \langle u_\ell^*, [u_i,u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*) X(u_\ell) - \sum_{i,j\in S_p^-} \varepsilon(u_i^*)([u_j,u_i] + \lambda k(u_i \mid u_j)_p)\varepsilon(u_j^*) \\ &\quad - \sum_{i,j\in S_p^-} \varepsilon(u_i^*)(e \mid [u_j,u_i]_p)\varepsilon(u_j^*) \\ &= -\frac{1}{2}\sum_{i,j,\ell\in S_p^-} c_{i,j}^\ell \varepsilon(u_i^*)\varepsilon(u_j^*) X(u_\ell) + \sum_{i,j,\ell\in S_p^-} c_{i,j}^\ell \varepsilon(u_i^*)\varepsilon(u_j^*)(u_\ell + (e \mid u_\ell)_p) \\ &= \frac{1}{2}\sum_{i,j,\ell\in S_p^-} c_{i,j}^\ell \varepsilon(u_i^*)\varepsilon(u_j^*) X(u_\ell). \end{aligned}$$

In the above calculations, we used the fact that $(u_i \mid u_j)_p = 0$ for $i,j\in S_p^-$. Moreover, when $c_{i,j}^\ell \neq 0$ and $i,j\in S_p^-$, we have $u_\ell \in \bigoplus_{i\leq -2} \mathfrak{g}_p(i)$, hence $\epsilon(u_\ell) = 0$ and $X(u_\ell) = u_\ell + (e \mid u_\ell)_p$.

For the other part, we have $\{\bar{d}^p{}_\lambda \bar{d}^{p,2}\} = A + B + C$, where

$$A = -\frac{1}{2} \sum_{i,j \in S_p^-} \{\bar{d}^p_\lambda \iota([u_i, u_j])\} \varepsilon(u_i^*) \varepsilon(u_j^*),$$

$$B = \frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j]) \{\bar{d}^p_\lambda \varepsilon(u_i^*)\} \varepsilon(u_j^*),$$

$$C = -\frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j]) \varepsilon(u_i^*) \{\bar{d}^p_\lambda \varepsilon(u_j^*)\}.$$

By the formulas in Lemma 4.3.1, we have

$$\begin{aligned} A &= -\frac{1}{2} \sum_{i,j,s,t \in S_p^-} \langle u_s^*, [u_i, u_j] \rangle \left(\iota([u_s, u_t]) \varepsilon(u_t^*) + X([u_i, u_j]) \right) \varepsilon(u_i^*) \varepsilon(u_j^*) \\ &= -\frac{1}{2} \sum_{i,j,s,t,\ell \in S_p^-} c_{i,j}^s \left(c_{s,t}^\ell \iota(u_\ell) \varepsilon(u_t^*) + X(u_s) \right) \varepsilon(u_i^*) \varepsilon(u_j^*) \\ &= -\frac{1}{2} \sum_{i,j,s,t,\ell \in S_p^-} c_{i,j}^s c_{s,t}^\ell \iota(u_\ell) \varepsilon(u_t^*) \varepsilon(u_i^*) \varepsilon(u_j^*) - \frac{1}{2} \sum_{i,j,s \in S_p^-} c_{i,j}^s X(u_s) \varepsilon(u_i^*) \varepsilon(u_j^*). \end{aligned}$$

By the formulas in Lemma 4.3.1, $\varepsilon(u_i^*)$ and $\{\bar{d}^p{}_\lambda \varepsilon(u_i^*)\}$ commute with each other, so

$$\begin{aligned} B + C &= \sum_{i,j \in S_p^-} \iota([u_i, u_j]) \{\bar{d}^p_\lambda \varepsilon(u_i^*)\} \varepsilon(u_j^*) \\ &= -\frac{1}{2} \sum_{i,j,s,t \in S_p^-} \iota([u_i, u_j]) \langle u_i^*, [u_s, u_t] \rangle \varepsilon(u_s^*) \varepsilon(u_t^*) \varepsilon(u_j^*) \\ &= \frac{1}{2} \sum_{i,j,s,t,\ell \in S_p^-} c_{i,j}^\ell c_{s,t}^i \iota(u_\ell) \varepsilon(u_s^*) \varepsilon(u_t^*) \varepsilon(u_j^*). \end{aligned}$$

Now it is clear that

$$A + B + C = -\frac{1}{2} \sum_{i,j,s \in S_p^-} c_{i,j}^s X(u_s) \varepsilon(u_i^*) \varepsilon(u_j^*).$$

Hence we have $\{\bar{d}^p{}_\lambda \bar{d}^p\} = 0$ and $(\bar{d}^p_0)^2 = 0$.

□

Definition 4.3.3. The *classical affine W-algebra* $\mathcal{W}^k(\mathfrak{g}_p, e)$ associated to the data $(\mathfrak{g}_p, e, k)$ is defined to be the homology $H(S(R^k(\mathfrak{g}_p, e)), \bar{d}^p_0)$. It inherits a Poisson vertex superalgebra structure from that of $S(R^k(\mathfrak{g}_p, e))$.

Remark 4.3.4. *Classical affine W-algebras ($p = 0$ and e regular case) were first discovered by Drinfeld and Sokolov [DS84]. They were used to construct hierarchies on some infinite-dimensional Poisson manifolds. P. Casati [Cas11] generalized Drinfeld and Sokolov's method and constructed hierarchies on affinizations of truncated current Lie algebras. The authors of [DSKV13] constructed more* *general integrable hierarchies corresponding to other classical affine W-algebras ($p = 0$ case). The classicial affine W-algebra $\mathcal{W}^k(\mathfrak{g}_p, e)$ might be used to construct integrable hierarchies on Drinfeld-Sokolov reductions of affinizations of truncated current Lie algebras.*

4.3.2 Quantum affine W-algebras as quantum Drinfeld-Sokolov reductions

Let $C^k(\mathfrak{g}_p, e) := V^k(\mathfrak{g}_p) \otimes F(A_p^{ch}) \otimes F(A_p^{ne})$ be the tensor product of the three vertex superalgebras, which is again a vertex superalgebra and contains $V^k(\mathfrak{g}_p)$, $F(A_p^{ch})$ and $F(A_p^{ne})$ as vertex subalgebras. Indeed, $C^k(\mathfrak{g}_p, e)$ can also be considered as the universal enveloping vertex superalgebra of $R^k(\mathfrak{g}_p, e)$.

Let

$$d^p = \sum_{i \in S_p^-} \varepsilon(u_i^*)(u_i + \epsilon(u_i) + (e \mid u_i)_p) - \frac{1}{2} \sum_{i,j \in S_p^-} : \iota([u_i, u_j])\varepsilon(u_i^*)\varepsilon(u_j^*) : . \tag{4.26}$$

The element $d^p \in C^k(\mathfrak{g}_p, e)$ is obviously odd. Define a charge grading on $C^k(\mathfrak{g}_p, e)$ by setting

$$\operatorname{cdeg} V^k(\mathfrak{g}_p) = \operatorname{cdeg} F(A_p^{ne}) = 0, \quad -\operatorname{cdeg} \iota(u) = \operatorname{cdeg} \varepsilon(v^*) = 1 \text{ for } u \in \mathfrak{n}_p, v^* \in \mathfrak{n}_p^*.$$

Then it induces a $\mathbb{Z}$-grading on $C^k(\mathfrak{g}_p, e)$ and d^p is of charge degree 1.

Lemma 4.3.5. *We have the following formulas for the λ-bracket of d^p in $C^k(\mathfrak{g}_p, e)$,*

(1) $[d^p{}_\lambda x] = \sum_{i \in S_p^-}([u_i, x] + k(x \mid u_i)_p(\lambda + T))\varepsilon(u_i^*)$ *for $x \in \mathfrak{g}_p$;*

(2) $[d^p{}_\lambda \epsilon(y)] = \sum_{i \in S_p^-}(e \mid [u_i, y])_p \varepsilon(u_i^*)$ *for $y \in \mathfrak{g}_p(-1)$;*

(3) $[d^p{}_\lambda \varepsilon(v^*)] = -\frac{1}{2} \sum_{i,j \in S_p^-} \langle v^*, [u_i, u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*)$ *for $v^* \in \mathfrak{n}_p^*$;*

(4) $[d^p{}_\lambda \iota(u)] = \sum_{i,j \in S_p^-} \langle u_i^*, u\rangle \Big(: \iota([u_i, u_j])\varepsilon(u_j^*) : + u_i + (e \mid u_i)_p + \epsilon(u_i)\Big)$ *for $u \in \mathfrak{n}_p$.*

Proof. The proof is the same as that of Lemma 4.3.1, except that we have (4.10) instead of (4.12). But we only need to notice two facts. First, the elements of $R^k(\mathfrak{g}_p, e)$ have same λ-bracket in $C^k(\mathfrak{g}_p, e)$ and in $S(R^k(\mathfrak{g}_p, e))$. Second, the extra term in (4.10) always vanishes in the calculations that we did in the proof of Lemma 4.3.1. Therefore, we have the same result in the end. □

Proposition 4.3.6. *We have $[d^p{}_\lambda d^p] = 0$, which implies that $(d_0^p)^2 = 0$ as d^p is odd.*

Proof. The proof is the same as that of Proposition 4.3.2 for the same reason as in the proof of Lemma 4.3.5. □

Definition 4.3.7. The *quantum affine W-algebra* $W^k(\mathfrak{g}_p, e)$ associated to the data $(\mathfrak{g}_p, e, k)$ is defined to be the cohomology of the complex $(C^k(\mathfrak{g}_p, e), d_0^p)$.

Remark 4.3.8. *When $p = 0$, the complex $(C^k(\mathfrak{g}_p, e), d_0^p)$ is called the BRST complex of the quantum Drinfeld-Sokolov reduction.*

4.3.3 Quantum affine W-algebra as (adjusted) semi-infinite cohomology

Let $\hat{\mathfrak{n}}_p = \mathfrak{n}_p \otimes \mathbb{C}[t, t^{-1}]$ be the affinization of $\mathfrak{n}_p$ with the $\mathbb{Z}$-grading: $(\hat{\mathfrak{n}}_p)_i = \mathfrak{n}_p \otimes t^i$. Denote by $u_{i,n} = u_i \otimes t^n$ for $u_i \in \mathfrak{n}_p$. Then $\{u_{i,n}\}_{n\in\mathbb{Z}, i\in S_p^-}$ forms a basis of $\hat{\mathfrak{n}}_p$. Let $\hat{\mathfrak{n}}_p^* := \mathfrak{n}_p^* \otimes \mathbb{C}[t, t^{-1}]$ and write $u_{i,n}^* = u_i^* \otimes t^n$. Then $\{u_{i,n}^*\}_{n\in\mathbb{Z}, i\in S_p^-}$ forms a basis of $\hat{\mathfrak{n}}_p^*$. One can identify $\hat{\mathfrak{n}}_p^*$ with the restricted dual of $\hat{\mathfrak{n}}_p$ under the pairing $\langle u_{j,m}^*, u_{i,n}\rangle := \delta_{m,-n-1}\delta_{i,j}$, and we define $(\hat{\mathfrak{n}}_p)_i^* = \mathfrak{n}_p^* \otimes t^{-i-1}$.

As $\mathfrak{n}_p$ is nilpotent, by Proposition 3.2.13, $\hat{\mathfrak{n}}_p$ admits a semi-infinite structure through

$$\rho^0(x) = \sum_{i\in S_p^-, n\in\mathbb{Z}} :\iota(\operatorname{ad} x(u_{i,n}))\varepsilon(u_{i,-n-1}^*): .$$

Let $\beta_e \in \hat{\mathfrak{n}}_p^*$ be defined by $\beta_e(u \otimes t^n) := \delta_{n,-1}(e \mid u)_p$ for $u \in \mathfrak{n}_p$. Then $\beta_e \in (\hat{\mathfrak{n}}_p^*)_{-1}$. Let $\rho^{\beta_e}(x) := \rho^0(x) + \beta_e(x)$ for $x \in \hat{\mathfrak{n}}_p$. Then for $x, y \in \hat{\mathfrak{n}}_p$,

$$\gamma^{\beta_e}(x, y) = [\rho^{\beta_e}(x), \rho^{\beta_e}(y)] - \rho^{\beta_e}([x, y]) = -\beta_e([x, y]).$$

Therefore, $\rho^{\beta_e}(x)$ gives a semi-infinite structure on $\hat{\mathfrak{n}}_p$ if and only if $\beta_e([x, y]) = 0$ for all $x, y \in \mathfrak{n}_p$, which is true if and only if the $\mathbb{Z}$-grading (4.20) is even.

Let $\mathfrak{m}_p = \bigoplus_{i\leq -2} \mathfrak{g}_p(i)$, $\hat{\mathfrak{m}}_p = \mathfrak{m}_p \otimes \mathbb{C}[t, t^{-1}]$ and $\hat{\mathfrak{g}}_p(-1) = \mathfrak{g}_p(-1) \otimes \mathbb{C}[t, t^{-1}]$. Note that $\ker \gamma^{\beta_e} = \hat{\mathfrak{m}}_p$, and $\hat{\mathfrak{g}}_p(-1)$ is a graded complement of $\ker \gamma^{\beta_e}$ in $\hat{\mathfrak{n}}_p$. Moreover, we have $[\hat{\mathfrak{n}}_p, \hat{\mathfrak{n}}_p] \subseteq \hat{\mathfrak{m}}_p$, so the Lie algebra $\hat{\mathfrak{n}}_p$ satisfies the assumption after Remark 3.3.2 with respect to the 1-cochain β_e in the discussion of adjusted semi-infinite cohomology. We can thus consider the adjusted semi-infinite cohomology of $\hat{\mathfrak{n}}_p$ with coefficients in the smooth module $V^k(\mathfrak{g}_p)$, with respect to β_e. The complex is $V^k(\mathfrak{g}_p) \otimes \Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^* \otimes \mathfrak{F}_{\beta_e}$ and the differential is

$$d^{\beta_e} = \sum_{i\in S_p^-, n\in\mathbb{Z}} u_{i,n}\varepsilon(u_{i,-n-1}^*) - \frac{1}{2} \sum_{\substack{i,j\in S_p^-,\\ n,m\in\mathbb{Z}}} :\iota([u_{i,n}, u_{j,m}])\varepsilon(u_{i,-n-1}^*)\varepsilon(u_{j,-m-1}^*): \\ + \varepsilon(\beta_e) + \sum_{i\in S_p', n\in\mathbb{Z}} \varepsilon(u_{i,-n-1}^*)\epsilon(u_{i,n}).$$

Recall the Lie (super)algebra structures on $cl(\hat{\mathfrak{n}}_p)$ and $\epsilon(\hat{\mathfrak{g}}_p(-1)) \oplus \mathbb{C}K$ defined in Chapter 3. Note that $\Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^*$ and $\mathfrak{F}_{\beta_e}$ are the Fock representations of $cl(\hat{\mathfrak{n}}_p)$ and $\epsilon(\hat{\mathfrak{g}}_p(-1)) \oplus \mathbb{C}K$, respectively. Since $cl(\hat{\mathfrak{n}}_p) \cong \widehat{A}_p^{ch}$ and $\epsilon(\hat{\mathfrak{g}}_p(-1)) \oplus \mathbb{C}K \cong \widehat{A}_p^{ne}$, we have $F(A_p^{ch}) \cong \Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^*$ and $F(A_p^{ne}) \cong \mathfrak{F}_{\beta_e}$ as vector spaces. Therefore, on the complex level, we have $C^k(\mathfrak{g}_p, e) \cong V^k(\mathfrak{g}_p) \otimes \Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^* \otimes \mathfrak{F}_{\beta_e}$. On the level of differential, $d_0^p = \operatorname{Res}_z d^p(z)$ is the coefficient of z^{-1} in the expression of $d^p(z)$, where $d^p(z)$ is the vertex operator of d^p defined by (4.26) and has the following form,

$$d^p(z) = \sum_{i\in S_p^-} u_i(z)\varepsilon(u_i^*)(z) - \frac{1}{2} \sum_{i,j\in S_p^-} :\iota([u_i, u_j])(z)\varepsilon(u_i^*)(z)\varepsilon(u_j^*)(z): \\ + \sum_{i\in S_p^-} (e \mid u_i)\varepsilon(u_i^*)(z) + \sum_{i\in S_p'} \varepsilon(u_i^*)(z)\epsilon(u_i)(z).$$

Recall the expressions of the vertex operators $u_i(z), \varepsilon(u_i^*)(z), \iota(u_i)(z)$ and $\epsilon(u_i)(z)$ given in the examples (4.2.12) and (4.2.14). We have

$$\mathrm{Res}_z \sum_{i\in S_p^-} u_i(z)\varepsilon(u_i^*)(z) = \mathrm{Res}_z \sum_{\substack{i\in S_p^-,\\ m,n\in\mathbb{Z}}} u_{i,n}\varepsilon(u_{i,m}^*)z^{-n-m-2} = \sum_{i\in S_p^-,n\in\mathbb{Z}} u_{i,n}\varepsilon(u_{i,-n-1}^*),$$

$$\mathrm{Res}_z \sum_{i\in S_p^-} (e \mid u_i)\varepsilon(u_i^*)(z) = \mathrm{Res}_z \sum_{i\in S_p^-,n\in Z} (e \mid u_i)_p\varepsilon(u_{i,n}^*)z^{-n-1} = \sum_{i\in S_p^-} (e \mid u_i)_p\varepsilon(u_{i,0}^*) = \varepsilon(\beta_e),$$

and

$$\mathrm{Res}_z \sum_{i\in S_p'} \varepsilon(u_i^*)(z)\epsilon(u_i)(z) = \mathrm{Res}_z \sum_{\substack{i\in S_p^-,\\ m,n\in\mathbb{Z}}} \varepsilon(u_{i,m}^*)\epsilon(u_{i,n})z^{-n-m-2} = \sum_{i\in S_p',n\in\mathbb{Z}} \varepsilon(u_{i,-n-1}^*)\epsilon(u_{i,n}).$$

Let $X = \sum_{i,j\in S_p^-} :\iota([u_i,u_j])(z)\varepsilon(u_i^*)(z)\varepsilon(u_j^*)(z):$. Then

$$\begin{aligned}\mathrm{Res}_z X &= \mathrm{Res}_z \sum_{\substack{i,j\in S_p^-,\\ n,m,\ell\in\mathbb{Z}}} :\iota([u_i,u_j]\otimes t^\ell)\varepsilon(u_{i,-m-1}^*)\varepsilon(u_{j,-n-1}^*): z^{m+n-\ell-1}\\ &= \sum_{\substack{i,j\in S_p^-,\\ n,m\in\mathbb{Z}}} :\iota([u_{i,n},u_{j,m}])\varepsilon(u_{i,-n-1}^*)\varepsilon(u_{j,-m-1}^*): .\end{aligned}$$

Therefore, we have $d_0^p = d^{\beta_e}$. This proves the following theorem.

Theorem 4.3.9 ([He17a]). *The affine W-algebra $W^k(\mathfrak{g}_p, e)$ is the adjusted semi-infinite cohomology of $\hat{\mathfrak{n}}_p$ with coefficients in $V^k(\mathfrak{g}_p)$ with respect to β_e. By Proposition 3.3.12, it is also an ordinary semi-infinite cohomology, i.e.,*

$$W^k(\mathfrak{g}_p, e) = H_a^{\infty/2+\bullet}(\hat{\mathfrak{n}}_p, \beta_e, V^k(\mathfrak{g}_p)) \cong H^{\infty/2+\bullet}(\hat{\mathfrak{n}}_p, V^k(\mathfrak{g}_p)\otimes\mathfrak{F}_{\beta_e})$$

Remark 4.3.10. *When the $\mathbb{Z}$-grading in (4.20) is even and $p = 0$, i.e., when ρ^{β_e} gives a semi-infinite structure on $\hat{\mathfrak{n}}_p$, the Fock representation $\mathfrak{F}_{\beta_e}$ reduces to a one-dimensional module $\mathbb{C}_{\beta_e}$ on which $x \in \hat{\mathfrak{n}}_p$ acts as $\beta_e(x)$. This recovers the semi-infinite cohomology realization of affine W-algebras in principal nilpotent cases [FF90].*

Remark 4.3.11. *When $p = 0$, the isomorphism $W^k(\mathfrak{g}_p, e) \cong H^{\infty/2+\bullet}(\hat{\mathfrak{n}}_p, V^k(\mathfrak{g}_p)\otimes\mathfrak{F}_{\beta_e})$ was also observed in [Ara05] (Remark 3.6.1), though the construction there was a bit different from ours.*