Chapter 2

Finite W-algebras associated to truncated current Lie algebras

In this chapter, we define finite W-algebras associated to truncated current Lie algebras and study some of their properties.

2.1 Truncated current Lie algebras

Given a finite-dimensional Lie algebra $\mathfrak{a}$, the *current algebra* associated to $\mathfrak{a}$ is the Lie algebra $\mathfrak{a}\otimes\mathbb{C}[t]$ with Lie bracket defined by $[a\otimes t^m, b\otimes t^n] := [a,b]\otimes t^{m+n}$ for $a,b\in\mathfrak{a}, m,n\in\mathbb{Z}_{\geq 0}$. One can show that the subspace $\mathfrak{a}\otimes t^p\mathbb{C}[t]$ is an ideal of $\mathfrak{a}\otimes\mathbb{C}[t]$ for any nonnegative integer p.

Definition 2.1.1. The *level p truncated current Lie algebra* associated to $\mathfrak{a}$ is the quotient Lie algebra

$$\mathfrak{a}_p := \frac{\mathfrak{a}\otimes\mathbb{C}[t]}{\mathfrak{a}\otimes t^{p+1}\mathbb{C}[t]} \cong \mathfrak{a}\otimes\frac{\mathbb{C}[t]}{t^{p+1}\mathbb{C}[t]}.$$

The Lie bracket of $\mathfrak{a}_p$ is

$$[a\otimes t^i, b\otimes t^j] = [a,b]\otimes t^{i+j}, \qquad \text{where } t^{i+j}\equiv 0 \text{ when } i+j>p.$$

Remark 2.1.2. *In the language of jet schemes [Mus01], $\mathfrak{a}_p$ is the p-th jet scheme of $\mathfrak{a}$. Truncated current Lie algebras are also called generalized Takiff algebras or polynomial Lie algebras.*

For convenience, we write xt^i for $x\otimes t^i$. An element of $\mathfrak{a}_p$ can be uniquely expressed as a sum $\sum_{i=0}^p x_it^i$ with $x_i\in\mathfrak{a}$. When $q\geq p$, the canonical surjective map $\pi_{q,p}:\mathfrak{a}_q\twoheadrightarrow\mathfrak{a}_p$ sending $\mathfrak{a}\otimes t^k$ to zero for $k\geq p+1$ is a Lie algebra homomorphism. For a subspace $\mathfrak{b}\subseteq\mathfrak{a}$, we let $\mathfrak{b}_p=\mathfrak{b}\otimes\frac{\mathbb{C}[t]}{t^{p+1}\mathbb{C}[t]}$, which is a subspace of $\mathfrak{a}_p$. If $\mathfrak{b}$ is a subalgebra of $\mathfrak{a}$, then $\mathfrak{b}_p$ is a subalgebra of $\mathfrak{a}_p$. For a nonnegative integer $k\leq p$, we denote by $\mathfrak{a}^{(k)}=\mathfrak{a}\otimes t^k$. By $\mathfrak{a}^{(\geq 1)}$ we mean $\bigoplus_{k\geq 1}\mathfrak{a}^{(k)}$. Then $\mathfrak{a}^{(0)}\cong\mathfrak{a}$ is a subalgebra of $\mathfrak{a}_p$ and $\mathfrak{a}^{(\geq 1)}$ is an ideal of $\mathfrak{a}_p$.

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Let $(\cdot \mid \cdot)$ be a symmetric bilinear form on $\mathfrak{a}$. Let $\bar{c} := (c_0, \cdots, c_p)$ with $c_i \in \mathbb{C}$. Define a symmetric bilinear form on $\mathfrak{a}_p$ by the formula

$$(x \mid y)_p := \sum_{k=0}^{p} c_k \sum_{i+j=k} (x_i \mid y_j), \tag{2.1}$$

where $x = \sum_{i=0}^{p} x_i t^i$ and $y = \sum_{i=0}^{p} y_i t^i$ with $x_i, y_i \in \mathfrak{a}$.

Lemma 2.1.3 ([Cas11])**.** *Assume that $(\cdot \mid \cdot)$ is non-degenerate and invariant on $\mathfrak{a}$. Then the bilinear form $(\cdot \mid \cdot)_p$ defined by (2.1) is invariant and symmetric. It is non-degenerate if and only if $c_p \neq 0$.*

Proof. Let $x = \sum_i x_i t^i, y = \sum_i y_i t^i$ and $z = \sum_i z_i t^i$ with $x_i, y_i, z_i \in \mathfrak{a}$. For the invariance, we have

$$\begin{aligned}([x, y] \mid z)_p &= \sum_{i,j,k} c_k([x_i, y_j] \mid z_{k-i-j}) \\ &= \sum_{i,j,k} c_k(x_i \mid [y_j, z_{k-i-j}]) \\ &= \sum_{i',j,k} c_k(x_{k-j-i'} \mid [y_j, z_{i'}]) \\ &= (x \mid [y, z])_p.\end{aligned}$$

If $c_p = 0$, it is clear that $\mathfrak{a}^{(p)}$ lies in the kernel of the form $(\cdot \mid \cdot)_p$, so it is degenerate. When $c_p \neq 0$, assume that $a = \sum_{i \geq i_0} a_i t^i$, with $a_{i_0} \neq 0$. By the non-degenerancy of $(\cdot \mid \cdot)$, there exists an element $b \in \mathfrak{a}$, such that $(a_{i_0} \mid b) \neq 0$. Then $(a \mid bt^{p-i_0})_p = c_p(a_{i_0} \mid b) \neq 0$, i.e., $(\cdot \mid \cdot)_p$ is non-degenerate.

□

Lemma 2.1.4. $\operatorname{Der} \dfrac{\mathbb{C}[t]}{\langle t^{p+1} \rangle} \cong \dfrac{t\mathbb{C}[t]}{\langle t^{p+1} \rangle} \dfrac{d}{dt}$.

Proof. Given a polynomial $f(t) \in t\mathbb{C}[t]/\langle t^{p+1} \rangle$, setting $g(t) \mapsto f(t)\dfrac{d}{dt}g(t)$ defines a derivation of $\mathbb{C}[t]/\langle t^{p+1} \rangle$. Conversely, let D be a derivation of $\mathbb{C}[t]/\langle t^{p+1} \rangle$. As $\mathbb{C}[t]/\langle t^{p+1} \rangle$ is generated by $\{1, t\}$ and $D(1) = 0$, D is determined by $D(t)$. Assume that $D(t) = g(t)$ for some $g(t) \in \mathbb{C}[t]/\langle t^{p+1} \rangle$. Then Leibniz's rule implies that $D(t^k) = kt^{k-1}g(t)$, i.e., $D = g(t)\dfrac{d}{dt}$. But $(p+1)t^p g(t) = D(t^{p+1}) = 0$ implies that $g(0) = 0$, so $g(t) \in t\mathbb{C}[t]/\langle t^{p+1} \rangle$ and $D \in t\mathbb{C}[t]/\langle t^{p+1} \rangle \dfrac{d}{dt}$. □

Let M be a $\mathfrak{g}$-module. A derivation from $\mathfrak{g}$ to M is a linear map $f : \mathfrak{g} \to M$ satisfying

$$f([a, b]) = a \cdot f(b) - b \cdot f(a) \text{ for all } a, b \in \mathfrak{g}.$$

The derivations from $\mathfrak{g}$ to M is denoted by $\operatorname{Der}(\mathfrak{g}, M)$. Given an element $m \in M$, define $\operatorname{ad} m(x) = x \cdot m$ for all $x \in \mathfrak{g}$. Then the Lie algebra action of $\mathfrak{g}$ on M implies that $\operatorname{ad} m \in \operatorname{Der}(\mathfrak{g}, M)$. Such derivations are called inner derivations and are denoted by $\operatorname{Inn}(\mathfrak{g}, M)$. We have $\operatorname{Der} \mathfrak{g} = \operatorname{Der}(\mathfrak{g}, \mathfrak{g})$ and $\operatorname{Inn} \mathfrak{g} = \operatorname{Inn}(\mathfrak{g}, \mathfrak{g})$, where $\mathfrak{g}$ is considered as the adjoint module of $\mathfrak{g}$.

In the language of Lie algebra cohomology (see Section 3.1), a derivation from $\mathfrak{g}$ to M is a 1-cocycle with coefficients in M and an inner derivation from $\mathfrak{g}$ to M is a 1-coboundary with coefficients in M, so $H^1(\mathfrak{g}, M) = \mathrm{Der}(\mathfrak{g}, M)/\mathrm{Inn}(\mathfrak{g}, M)$.

Lemma 2.1.5 (Whitehead). *Let $\mathfrak{g}$ be finite-dimensional semi-simple Lie algebra and M a finite-dimensional non-trivial simple $\mathfrak{g}$-module. Then $H^i(\mathfrak{g}, M) = 0$ for all $i > 0$, in particular, we have* $\mathrm{Der}(\mathfrak{g}, M) = \mathrm{Inn}(\mathfrak{g}, M)$.

Let $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ and $d \in \mathrm{Der}\, \dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$. Consider the map $D = \varphi \otimes d : \mathfrak{g}_p \to \mathfrak{g}_p$ defined by sending $a \otimes f(t)$ to $\varphi(a) \otimes df(t)$. We have

$$D([a \otimes f(t), b \otimes g(t)]) = D([a, b] \otimes f(t)g(t)) = \varphi([a, b]) \otimes d(f(t)g(t)).$$

Since $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$, we have $\varphi([a, b]) = [a, \varphi(b)] = -\varphi([b, a]) = -[b, \varphi(a)] = [\varphi(a), b]$. Since $d \in \mathrm{Der}\, \dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$, we have $d(f(t)g(t)) = d(f(t))g(t) + f(t)d(g(t))$. Therefore, we have

$$\begin{aligned} D([a \otimes f(t), b \otimes g(t)]) &= \varphi([a, b]) \otimes (d(f(t))g(t) + f(t)d(g(t))) \\ &= [\varphi(a), b] \otimes d(f(t))g(t) + [a, \varphi(b)] \otimes f(t)d(g(t)) \\ &= [D(a \otimes f(t)), b \otimes g(t)] + [a \otimes f(t), D(b \otimes g(t))], \end{aligned}$$

i.e., $\varphi \otimes d \in \mathrm{Der}\, \mathfrak{g}_p$.

Proposition 2.1.6. *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra. Then*

$$\mathrm{Der}\, \mathfrak{g}_p \cong \left(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der} \frac{\mathbb{C}[t]}{\langle t^{p+1}\rangle} \right) \ltimes \mathrm{Inn}\, \mathfrak{g}_p.$$

Proof. Given $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ and $d \in \mathrm{Der}\, \dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$, we have $(\varphi \otimes d)(\mathfrak{g}^{(0)}) = 0$, so every element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der} \dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$ kills $\mathfrak{g}^{(0)}$. But we have $\mathrm{ad}\, x(\mathfrak{g}^{(0)}) \neq 0$ for all $x \in \mathfrak{g}_p$ which is non-zero, so

$$\mathrm{Inn}\, \mathfrak{g}_p \cap \left(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der} \frac{\mathbb{C}[t]}{\langle t^{p+1}\rangle} \right) = 0.$$

We know that $\mathrm{Inn}\, \mathfrak{g}_p$ is an ideal of $\mathrm{Der}\, \mathfrak{g}_p$, so we only need to prove that

$$\mathrm{Der}\, \mathfrak{g}_p = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der} \frac{\mathbb{C}[t]}{\langle t^{p+1}\rangle} + \mathrm{Inn}\, \mathfrak{g}_p.$$

For $0 \leq i \leq p$, let π_i be the projection of $\mathfrak{g}_p$ to the subspace $\mathfrak{g}^{(i)}$, i.e., $\pi_i(\sum_{k=0}^{p} x_k t^k) = x_i t^i$.

Note that $\mathfrak{g}_p$ is generated by $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$, so a derivation $D \in \mathrm{Der}\, \mathfrak{g}_p$ is determined by its value on $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$. Let $D_i = \pi_i \circ D$. Then we have $D = \sum_{i=0}^{p} D_i$. Composing π_i with Leibniz's rule, we get

$$D_i([a \otimes 1, b \otimes 1]) = [D_i(a \otimes 1), b \otimes 1] + [a \otimes 1, D_i(b \otimes 1)].$$

That means, when restricted to $\mathfrak{g}^{(0)}$, $D_i \in \operatorname{Der}(\mathfrak{g}^{(0)}, \mathfrak{g}^{(i)})$. Since $\mathfrak{g}^{(0)} \cong \mathfrak{g}$ is semi-simple, we have $\operatorname{Der}(\mathfrak{g}^{(0)}, \mathfrak{g}^{(i)}) = \operatorname{Inn}(\mathfrak{g}^{(0)}, \mathfrak{g}^{(i)})$ by Lemma 2.1.5. Therefore, there exists $x_i \otimes t^i \in \mathfrak{g} \otimes t^i$ for each $0 \leq i \leq p$, such that $\operatorname{ad}(x_i \otimes t^i) = D_i$ when restricted to $\mathfrak{g}^{(0)}$. Let $D' = D - \sum_{i=0}^{p} \operatorname{ad}(x_i \otimes t^i)$. Then $D'|_{\mathfrak{g}^{(0)}} = 0$. Let $D'_i = \pi_i \circ D'$. Applying D' to $[a \otimes 1, b \otimes t]$ and composing with π_i, by Leibniz's rule, we get

$$D'_i([a \otimes 1, b \otimes t]) = [a \otimes 1, D'_i(b \otimes t)]. \tag{2.2}$$

When restricted to $\mathfrak{g}^{(1)}$, (2.2) implies that $D'_i : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(i)}$ is a $\mathfrak{g}^{(0)}$-module homomorphism. As $\mathfrak{g}^{(1)} \cong \mathfrak{g}^{(i)} \cong \mathfrak{g}$ as $\mathfrak{g}$-modules, there exist $\mathfrak{g}$-module homomorphisms $\varphi_i : \mathfrak{g} \to \mathfrak{g}$ such that $D'_i = \varphi_i \otimes t^{i-1}$ when restricted to $\mathfrak{g}^{(1)}$. Note that for $i \geq 1$, $D'_i = \varphi_i \otimes t^i \dfrac{d}{dt} \in \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \operatorname{Der} \dfrac{\mathbb{C}[t]}{\langle t^{p+1} \rangle}$, when D'_i is restricted to $\mathfrak{g}^{(1)}$. Let $D'' = D' - \sum_{i \geq 1}^{p} \varphi_i \otimes t^i \dfrac{d}{dt}$. Then $D''|_{\mathfrak{g}^{(0)}} = 0$ and $D''(\mathfrak{g}^{(1)}) \subseteq \mathfrak{g}^{(0)}$. We show that $D'' = 0$. Note that we have $D'' = D'_0 = \varphi_0 \otimes t^{-1}$ when restricted to $\mathfrak{g}^{(1)}$, where $\varphi_0 : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(0)}$ is a $\mathfrak{g}^{(0)}$-module homomorphism. By Leibniz's rule, we have

$$\begin{aligned} D''([a \otimes t, b \otimes t]) &= [D''(a \otimes t), b \otimes t] + [a \otimes t, D''(b \otimes t)] \\ &= [\varphi_0(a), b] \otimes t + [a, \varphi_0(b)] \otimes t \\ &= \varphi_0[a, b] \otimes 2t. \end{aligned}$$

Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have $D''(a \otimes t^2) = \varphi_0(a) \otimes 2t$ for all $a \in \mathfrak{g}$. Inductively, we have $D''(a \otimes t^k) = \varphi_0(a) \otimes k t^{k-1}$. In particular, $D''(a \otimes t^{p+1}) = \varphi_0(a) \otimes p t^p$ for all $a \in \mathfrak{g}$. Since $a \otimes t^{p+1} = 0$ in $\mathfrak{g}_p$, we have $\varphi_0(a) = 0$ for all $a \in \mathfrak{g}$, i.e., $D'' = 0$, and

$$D = \sum_{i=1}^{p} \operatorname{ad}(x_i \otimes t^i) + \sum_{i \geq 1}^{p} \varphi_i \otimes t^i \frac{d}{dt} \in \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \operatorname{Der} \frac{\mathbb{C}[t]}{\langle t^{p+1} \rangle} + \operatorname{Inn} \mathfrak{g}_p.$$

□

2.2 Finite W-algebras via Whittaker model definition

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$ with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$. By Lemma 2.1.3, there exists a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)_p$ on $\mathfrak{g}_p$, which we fix from now on.

Let $\Gamma : \mathfrak{g} \overset{\operatorname{ad} h_\Gamma}{=\!=} \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be a good $\mathbb{Z}$-grading of $\mathfrak{g}$ with a good element $e \in \mathfrak{g}(2)$, and $\{e, f, h\}$ an $s\ell_2$-triple containing e with $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-2)$. Let $\mathfrak{g}_p(i) := \{x \in \mathfrak{g}_p \mid [h_\Gamma, x] = ix\}$. Then $\Gamma_p : \mathfrak{g}_p = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_p(i)$ is a $\mathbb{Z}$-grading of $\mathfrak{g}_p$.

Lemma 2.2.1. *The $\mathbb{Z}$-grading Γ_p of $\mathfrak{g}_p$ is good with good element e.*

Proof. Note that $\mathfrak{g}_p(i) = \mathfrak{g}(i)_p$. For the map $\operatorname{ad} e : \mathfrak{g}_p(i) \to \mathfrak{g}_p(i+2)$, we have $\ker \operatorname{ad} e = (\mathfrak{g}(i)^e)_p$ and $\operatorname{im} \operatorname{ad} e = ([\mathfrak{g}(i), e])_p$, so it is injective for $i \leq -1$ and surjective for $i \geq -1$ as e is a good element with respect to Γ. □

Remark 2.2.2. *We call Γ_p a good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from a good $\mathbb{Z}$-grading of $\mathfrak{g}$.*

Example 2.2.3. In this example, we show that not every good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ is induced from a good $\mathbb{Z}$-grading of $\mathfrak{g}$ as in Lemma 2.2.1. Let $\mathfrak{g} = s\ell_2$ with canonical basis $\{e, f, h\}$ such that $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Consider $\mathfrak{g}_2$, which has a basis $\{e, f, h, e \otimes t, f \otimes t, h \otimes t\}$. Let $x = h + 2e \otimes t + 2f \otimes t$. Then with respect to $\operatorname{ad} x$, we have the $\mathbb{Z}$-grading on $\mathfrak{g}_2$

$$\mathfrak{g}_2 = \mathfrak{g}_2(-2) \oplus \mathfrak{g}_2(0) \oplus \mathfrak{g}_2(2) \tag{2.3}$$

with $\mathfrak{g}_2(-2) = \operatorname{span}_{\mathbb{C}}\{f \otimes t, f - h \otimes t\}, \mathfrak{g}_2(0) = \operatorname{span}_{\mathbb{C}}\{h \otimes t, h + 2e \otimes t + 2f \otimes t\}$, and $\mathfrak{g}_2(2) = \operatorname{span}_{\mathbb{C}}\{e \otimes t, e - h \otimes t\}$. It is easy to check that $e - h \otimes t$ is a good element with respect to (2.3).

Moreover, Jacobson-Morozov's lemma does not work in truncated current Lie algebras. Indeed, when $p \geq 1$, $x \otimes t$ is nilpotent in $\mathfrak{g}_p$ for any $x \in \mathfrak{g}$ and it cannot be embedded into any $s\ell_2$-triple.

Lemma 2.2.4. *Let $\Gamma_p : \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_p(i)$ be a $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from a good $\mathbb{Z}$-grading of $\mathfrak{g}$. We have $(\mathfrak{g}_p(i) \mid \mathfrak{g}_p(j))_p = 0$ if $i + j \neq 0$.*

Proof. Let h_Γ be the semi-simple element defining Γ_p. Let $x \in \mathfrak{g}_p(i), y \in \mathfrak{g}_p(j)$ and $i + j \neq 0$. Then $([h_\Gamma, x] \mid y)_p = -(x \mid [h_\Gamma, y])_p$, i.e., $(i+j)(x \mid y)_p = 0$. Since $i+j \neq 0$, that implies $(x \mid y)_p = 0$. $\square$

Let $\chi_p = (e \mid \cdot)_p \in \mathfrak{g}_p^*$. Define a skew-symmetric bilinear form on $\mathfrak{g}_p(-1)$ by

$$\langle \cdot, \cdot \rangle_p : \mathfrak{g}_p(-1) \times \mathfrak{g}_p(-1) \to \mathbb{C}, \qquad (x, y) \mapsto \langle x, y \rangle_p := \chi_p([x, y]). \tag{2.4}$$

Lemma 2.2.5. *The bilinear form on $\mathfrak{g}_p(-1)$ defined by (2.4) is non-degenerate.*

Proof. This follows from the surjectivity of $\operatorname{ad} e : \mathfrak{g}_p(-1) \to \mathfrak{g}_p(1)$, the invariance of the bilinear form $(\cdot \mid \cdot)_p$ and the pairing property $(\mathfrak{g}_p(i) \mid \mathfrak{g}_p(j))_p = 0$ if $i + j \neq 0$. $\square$

Let $\mathfrak{l}_p$ be an isotropic subspace of $\mathfrak{g}_p(-1)$ with respect to the bilinear form (2.4), i.e., $(e \mid [\mathfrak{l}_p, \mathfrak{l}_p])_p = 0$. Let $\mathfrak{l}_p^\perp := \{x \in \mathfrak{g}_p(-1) \mid (e \mid [x, y])_p = 0 \text{ for all } y \in \mathfrak{l}_p\}$, and let

$$\mathfrak{m}_p := \bigoplus_{i \leq -2} \mathfrak{g}_p(i), \qquad \mathfrak{m}_{\mathfrak{l},p} := \mathfrak{m}_p \oplus \mathfrak{l}_p, \qquad \mathfrak{n}_{\mathfrak{l},p} := \mathfrak{m}_p \oplus \mathfrak{l}_p^\perp, \qquad \mathfrak{n}_p := \bigoplus_{i \leq -1} \mathfrak{g}_p(i). \tag{2.5}$$

Obviously, $\mathfrak{m}_p \subseteq \mathfrak{m}_{\mathfrak{l},p} \subseteq \mathfrak{n}_{\mathfrak{l},p} \subseteq \mathfrak{n}_p$ are all nilpotent subalgebras of $\mathfrak{g}_p$.

One can easily show that $(e \mid [\mathfrak{m}_{\mathfrak{l},p}, \mathfrak{n}_{\mathfrak{l},p}])_p = 0$, thanks to the property $(e \mid \mathfrak{g}_p(i))_p = 0$ for $i \leq -3$ and the definition of $\mathfrak{l}_p$ and $\mathfrak{l}_p^\perp$. In particular, $\chi_p = (e \mid \cdot)_p$ is a character of $\mathfrak{m}_{\mathfrak{l},p}$ hence defines a one-dimensional representation of $\mathfrak{m}_{\mathfrak{l},p}$, which we denote by $\mathbb{C}_{\chi_p}$. Let

$$Q_{\chi_p} := U(\mathfrak{g}_p) \otimes_{U(\mathfrak{m}_{\mathfrak{l},p})} \mathbb{C}_{\chi_p} \cong U(\mathfrak{g}_p)/I_{\chi_p},$$

where I_{χ_p} is the left ideal of $U(\mathfrak{g}_p)$ generated by $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. We denote by $\bar{u} := u + I_{\chi_p}$ for the image of $u \in U(\mathfrak{g}_p)$ in Q_{χ_p}.

Lemma 2.2.6. *The adjoint action of $\mathfrak{n}_{\mathfrak{l},p}$ on $U(\mathfrak{g}_p)$ leaves the subspace I_{χ_p} invariant.*

Proof. Let $x \in \mathfrak{n}_{\mathfrak{l},p}$ and $y = \sum_i u_i(a_i - \chi_p(a_i)) \in I_{\chi_p}$, with $u_i \in U(\mathfrak{g}_p)$ and $a_i \in \mathfrak{m}_{\mathfrak{l},p}$. Then

$$\begin{aligned}[x, y] &= \sum_i [x, u_i(a_i - \chi_p(a_i))] \\ &= \sum_i \left([x, u_i](a_i - \chi_p(a_i)) + u_i[x, a_i - \chi_p(a_i)]\right).\end{aligned}$$

Since $\chi_p([\mathfrak{n}_{\mathfrak{l},p}, \mathfrak{m}_{\mathfrak{l},p}]) = 0$, we have $[x, a_i - \chi_p(a_i)] = [x, a_i] \in I_{\chi_p}$, hence $[x, y] \in I_{\chi_p}$. □

Since $\operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}$ preserves I_{χ_p}, it induces a well-defined adjoint action on Q_{χ_p}, such that

$$[x, \bar{u}] = \overline{[x, u]} \text{ for } x \in \mathfrak{n}_{\mathfrak{l},p}, u \in U(\mathfrak{g}_p).$$

Let

$$H_{\chi_p} := Q_{\chi_p}^{\operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}} = \{\bar{u} \in Q_{\chi_p} \mid [x, u] \in I_{\chi_p} \text{ for all } x \in \mathfrak{n}_{\mathfrak{l},p}\}.$$

Lemma 2.2.7. *There is a well-defined multiplication on H_{χ_p} by*

$$\bar{u} \cdot \bar{v} := \overline{uv} \text{ for } \bar{u}, \bar{v} \in H_{\chi_p}.$$

Proof. First, we show that the multiplication $\bar{u} \cdot \bar{v}$ does not depend on the representatives. It is obvious that it does not depend on the representatives of v. For that of u, we need to show that $yv \in I_{\chi_p}$ for all $y \in I_{\chi_p}, \bar{v} \in H_{\chi_p}$. Assume that $y = \sum_i u_i(a_i - \chi_p(a_i))$ with $a_i \in \mathfrak{m}_{\mathfrak{l},p}$, then

$$yv = [y, v] + vy = \sum_i u_i[a_i - \chi_p(a_i), v] + \sum_i [u_i, v](a_i - \chi_p(a_i)) + vy. \tag{2.6}$$

By the definition of H_{χ_p}, we have $[a_i + \chi_p(a_i), v] = [a_i, v] \in I_{\chi_p}$ since $a_i \in \mathfrak{m}_{\mathfrak{l},p} \subseteq \mathfrak{n}_{\mathfrak{l},p}$, hence $yv \in I_{\chi_p}$.

Next we show that H_{χ_p} is closed under the multiplication. Let $\bar{u}_1, \bar{u}_2 \in H_{\chi_p}$, we need show that $\overline{u_1 u_2} \in H_{\chi_p}$, i.e., $[x, u_1 u_2] \in I_{\chi_p}$ for all $x \in \mathfrak{n}_{\mathfrak{l},p}$. By Leibniz's rule, we have

$$[x, u_1 u_2] = [x, u_1]u_2 + u_1[x, u_2].$$

By the definition of H_{χ_p}, we have $[x, u_1], [x, u_2] \in I_{\chi_p}$. Therefore, $[x, u_1]u_2 \in I_{\chi_p}$ by (2.6). □

Once the multiplication is well-defined, H_{χ_p} inherits an associative algebra structure from $U(\mathfrak{g}_p)$.

Definition 2.2.8. The *finite W-algebra* $W^{fin}(\mathfrak{g}_p, e)$ associated to the pair $(\mathfrak{g}_p, e)$ is defined to be H_{χ_p}.

Remark 2.2.9. *When $p = 0$, we get the definition of the finite W-algebra associated to the semi-simple Lie algebra $\mathfrak{g}$ and the nilpotent element e given by A. Premet in [Pre02].*

When $\mathfrak{l}_p$ is a Lagrangian subspace, i.e., $\mathfrak{l}_p = \mathfrak{l}_p^{\perp}$ hence $\mathfrak{m}_{\mathfrak{l},p} = \mathfrak{n}_{\mathfrak{l},p}$, we can realize H_{χ_p} as the opposite endomorphism algebra $(\mathrm{End}_{U(\mathfrak{g}_p)} Q_{\chi_p})^{op}$ in the following way. As $Q_{\chi_p} = U(\mathfrak{g}_p)/I_{\chi_p}$ is a cyclic $\mathfrak{g}_p$-module, an endomorphism φ is determined by its value on the generator $\bar{1}$. Since $\bar{1}$ is killed by I_{χ_p}, $\varphi(\bar{1})$ must be annihilated by I_{χ_p}. On the other hand, given an element $\bar{y} \in Q_{\chi_p}$, which is killed by I_{χ_p}, $\bar{1} \mapsto \bar{y}$ defines an endomorphism of Q_{χ_p}. We thus have

$$\begin{aligned}(\mathrm{End}_{U(\mathfrak{g}_p)} Q_{\chi_p})^{op} &\cong \{\bar{y} \in Q_{\chi_p} \mid (a - \chi_p(a))y \in I_{\chi_p} \text{ for all } a \in \mathfrak{m}_{\mathfrak{l},p}\} \\ &= \{\bar{y} \in Q_{\chi_p} \mid [a, y] \in I_{\chi_p} \text{ for all } a \in \mathfrak{n}_{\mathfrak{l},p}\} \\ &= H_{\chi_p}.\end{aligned}$$

Remark 2.2.10. *When $p = 0$, it was proved that the finite W-algebras H_{χ_0} with respect to different good gradings Γ_0 [BG07] and different isotropic subspaces $\mathfrak{l}_0$ [GG02] are all isomorphic. For $p \geq 1$, we will show the independence of isotropic subspace $\mathfrak{l}_p$ in the sequel following [GG02].*

Remark 2.2.11. *As in the semi-simple case [BGK08], there are other definitions of finite W-algebras in the truncated current setting.*

2.3 Quantization of Slodowy slices

We keep the notation of Section 2.1 and Section 2.2.

2.3.1 Poisson structure on Slodowy slices

The non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)_p$ on $\mathfrak{g}_p$ defines a bijection $\kappa_p : \mathfrak{g}_p \to \mathfrak{g}_p^*$ through $x \mapsto (x \mid \cdot)_p$. Let $\mathfrak{g}_p^f$ be the centralizer of f in $\mathfrak{g}_p$. Set

$$\mathcal{S}_{e_p} := e + \mathfrak{g}_p^f \quad \text{and} \quad \mathcal{S}_{\chi_p} := \chi_p + \ker \mathrm{ad}^* f = \kappa_p(\mathcal{S}_{e_p}).$$

When $p = 0$, $\mathcal{S}_e := \mathcal{S}_{e_0}$ is called the *Slodowy slice* through e [Slo80]. In the language of jet schemes [Mus01], $\mathcal{S}_{e_p}$ is the p-th jet scheme of $\mathcal{S}_e$. We also call S_{e_p} the Slodowy slice through e in $\mathfrak{g}_p$ and S_{χ_p} the Slodowy slice through χ_p in $\mathfrak{g}_p^*$.

By the representation theory of $s\ell_2$, we have $\mathfrak{g}_p = \mathfrak{g}_p^e \oplus [\mathfrak{g}_p, f] = \mathfrak{g}_p^f \oplus [\mathfrak{g}_p, e]$, which implies that $\mathrm{ad}\, e : [f, \mathfrak{g}_p] \xrightarrow{1:1} [e, \mathfrak{g}_p]$ and $\mathrm{ad}\, f : [e, \mathfrak{g}_p] \xrightarrow{1:1} [f, \mathfrak{g}_p]$ are both bijective.

Lemma 2.3.1. *Let $r \in \bigoplus_{i \leq 1} \mathfrak{g}_p(i)$. Then*

(a) $[e + r, [f, \mathfrak{g}_p]] \cap \mathfrak{g}_p^f = 0$.

(b) The map $\mathrm{ad}\,(e + r) : [f, \mathfrak{g}_p] \to [e + r, [f, \mathfrak{g}_p]]$ *is bijective.*

(c) If $a \in \mathfrak{g}_p$ is such that $[e + r, a] \in \mathfrak{g}_p^f$ and $(a \mid [e + r, \mathfrak{g}_p] \cap \mathfrak{g}_p^f)_p = 0$, then $[e + r, a] = 0$.

(d) $[e + r, [f, \mathfrak{g}_p]] \oplus \mathfrak{g}_p^f = [e + r, \mathfrak{g}_p] + \mathfrak{g}_p^f = \mathfrak{g}_p$.

Proof. Let $a = \sum_i a_i$ with $a_i \in \mathfrak{g}_p(i)$ such that $[f, a] \neq 0$. Let i_0 be such that $[f, a_{i_0}] \neq 0$ but $[f, a_i] = 0$ for all $i > i_0$. Then the i_0-th component (which belongs to $\mathfrak{g}_p(i_0)$) of $[e + r, [f, a]]$ is $[e, [f, a_{i_0}]]$ as $r \in \bigoplus_{i \leq 1} \mathfrak{g}_p(i)$ and $e \in \mathfrak{g}_p(2)$. Since $[f, a_{i_0}] \neq 0$ and $\mathrm{ad}\, e : [f, \mathfrak{g}_p] \to [e, \mathfrak{g}_p]$ is bijective, we have $[e, [f, a_{i_0}]] \neq 0$.

(a) Assume $a \in \mathfrak{g}_p$ satisfies that $0 \neq [e + r, [f, a]] \in \mathfrak{g}_p^f$. Then $[f, a] \neq 0$. Let i_0 be as above, then $0 \neq [e, [f, a_{i_0}]] \in \mathfrak{g}_p^f(i_0)$ i.e., $[f, [e, [f, a_{i_0}]]] = 0$. This contradicts to the bijectivity of $\mathrm{ad}\, f : [e, \mathfrak{g}_p] \to [f, \mathfrak{g}_p]$.

(b) We just need to show that $\mathrm{ad}\,(e + r)$ is injective on $[f, \mathfrak{g}_p]$. Suppose that $[e + r, [f, a]] = 0$ with $[f, a] \neq 0$. Let i_0 be as above. Then its i_0-th component $[e, [f, a_{i_0}]] \neq 0$, a contradiction.

(c) For a subspace V of $\mathfrak{g}_p$, we denote by $V^\perp$ its orthogonal complement with respect to $(\cdot \mid \cdot)_p$. Then $([e + r, \mathfrak{g}_p] \cap \mathfrak{g}_p^f)^\perp = [e + r, \mathfrak{g}_p]^\perp + (\mathfrak{g}_p^f)^\perp$. Note that $(\mathfrak{g}_p^f)^\perp = [f, \mathfrak{g}_p]$ and $[e + r, \mathfrak{g}_p]^\perp = \ker \mathrm{ad}\,(e + r)$ as $(\cdot \mid \cdot)_p$ is non-degenerate and invariant. Therefore, (c) is equivalent to saying that if $a = u + v$ with $u \in (\mathfrak{g}_p^f)^\perp = [f, \mathfrak{g}_p], v \in [e+r, \mathfrak{g}_p]^\perp$ and $[e+r, a] \in \mathfrak{g}_p^f$, then $[e+r, a] = 0$. Since $u \in [f, \mathfrak{g}_p]$ and $v \in \ker \mathrm{ad}\,(e + r)$, we have $[e + r, a] = [e + r, u] \in \mathfrak{g}_p^f \cap [e + r, [f, \mathfrak{g}_p]]$, which must be zero by (a).

(d) It is enough to prove $[e + r, [f, \mathfrak{g}_p]] \oplus \mathfrak{g}_p^f = \mathfrak{g}_p$. It is a direct sum because of (a). Let us count dimensions. We have $\dim[e + r, [f, \mathfrak{g}_p]] = \dim[f, \mathfrak{g}_p]$ by (b). Note that $\dim \mathfrak{g}_p^f = \dim \mathfrak{g}_p^e$ and $\dim[f, \mathfrak{g}_p] = \dim \mathfrak{g}_p - \dim \mathfrak{g}_p^e$ as we have $\mathfrak{g}_p = [\mathfrak{g}_p, f] \oplus \mathfrak{g}_p^e$, so $\dim \mathfrak{g}_p = \dim \mathfrak{g}_p^f + \dim[f, \mathfrak{g}_p]$, and (d) is proved.

□

Remark 2.3.2. *Lemma 2.3.1 was proved in [DSKV16] for* $r \in \bigoplus_{i \leq 0} \mathfrak{g}(i)$ *and* $\mathfrak{g}$ *semi-simple, where* $\Gamma : \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ *is a good* $\mathbb{Z}$*-grading of* $\mathfrak{g}$ *with a good element* $e \in \mathfrak{g}(2)$. *We have used the same argument to prove the truncated current version above.*

Combining Theorem 1.1.9 and Lemma 2.3.1, we have the following lemma.

Lemma 2.3.3. *The slice* S_{e_p} *has a Poisson structure.*

Proof. We show that the two conditions in Theorem 1.1.9 are satisfied for the submanifold S_{e_p} of $\mathfrak{g}_p$. Let $x = e + r \in S_{e_p} \cap \mathbb{O}_x$, where $\mathbb{O}_x$ is the adjoint orbit of $\mathfrak{g}_p$ through x. As $r \in \bigoplus_{i \leq 0} \mathfrak{g}_p(i)$, Lemma 2.3.1 applies. Note that $T_x S_{e_p} = \mathfrak{g}_p^f$ and $T_x \mathbb{O}_x = [\mathfrak{g}_p, x]$. Part (d) of Lemma 2.3.1 shows that S_{e_p} is transversal to $\mathbb{O}_x$ at x. Next we show that the restriction of the symplectic form ω_x defined by (1.4) on the subspace $T_x \mathbb{O}_x \cap T_x S_{e_p} = [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ is non-degenerate. Assume that there exists an element $[a, x] \in [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ such that $[a, x] \in \ker \omega_x|_{[\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f}$, i.e.,

$$\omega_x([a, x], [b, x]) = (x \mid [a, b])_p = (a \mid [b, x])_p = 0$$

for all $[b, x] \in [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$. Part (c) of Lemma 2.3.1 shows that $[a, x] = 0$. Therefore, ω_x is non-degenerate when restricted to $[\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ and S_{e_p} inherits a Poisson structure from that of $\mathfrak{g}_p$. □

Corollary 2.3.4. The Slodowy slice $\mathcal{S}_{\chi_p}$ has a Poisson structure.

Definition 2.3.5. The *classical finite W-algebra* associated to $(\mathfrak{g}_p, e)$ is defined to be the Poisson algebra $\mathbb{C}[\mathcal{S}_{\chi_p}]$.

Remark 2.3.6. *Explicit formulas for the Poisson bracket of* $\mathbb{C}[\mathcal{S}_{\chi_p}]$ *were calculated in [DSKV16] for* $p = 0$.

2.3.2 An isomorphism of affine varieties

Let G_p be the adjoint group of $\mathfrak{g}_p$ and $N_{\mathfrak{l},p}$ the unipotent subgroup of G_p with Lie algebra $\mathfrak{n}_{\mathfrak{l},p}$. Let

$$\mathfrak{m}_{\mathfrak{l},p}^{\perp} := \{x \in \mathfrak{g}_p \mid (x \mid y)_p = 0 \text{ for all } y \in \mathfrak{m}_{\mathfrak{l},p}\}$$

be the orthogonal complement of $\mathfrak{m}_{\mathfrak{l},p}$ with respect to the bilinear form $(\cdot \mid \cdot)_p$. One can show that $\mathfrak{m}_{\mathfrak{l},p}^{\perp} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_p(i)\right) \oplus [\mathfrak{l}_p^{\perp}, e]$. As $\mathfrak{n}_{\mathfrak{l},p}$ is nilpotent, the subgroup $N_{\mathfrak{l},p}$ is generated by $\exp(\operatorname{ad} x)$ with x running through $\mathfrak{n}_{\mathfrak{l},p}$. Restrict the adjoint action of $N_{\mathfrak{l},p}$ to $e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Assume that $y \in \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Note that

$$\exp(\operatorname{ad} x)(e + y) = (1 + \operatorname{ad} x + \cdots + \frac{\operatorname{ad}^n x}{n!} + \cdots)(e + y) \in e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}.$$

Therefore, the image of the action map $N_{\mathfrak{l},p} \times (e + \mathfrak{m}_{\mathfrak{l},p}^{\perp})$ is contained in $e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Since $S_{e_p} \subseteq e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$, we can moreover restrict the adjoint action map to $N_{\mathfrak{l},p} \times S_{e_p}$. There is an $N_{\mathfrak{l},p}$-action on $N_{\mathfrak{l},p} \times S_{e_p}$ defined by $u \cdot (v, x) = (uv, x)$ for $u, v \in N_{\mathfrak{l},p}$ and $x \in S_{e_p}$. Note that

$$u \cdot (v, x) = (uv, x) = (uv) \cdot x = u \cdot (v \cdot x),$$

so the adjoint action map $N_{\mathfrak{l},p} \times S_{e_p} \to e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$ is $N_{\mathfrak{l},p}$-equivariant, where $N_{\mathfrak{l},p}$ acts on $e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$ by adjoint action.

Lemma 2.3.7. *The adjoint action map* $\beta : N_{\mathfrak{l},p} \times S_{e_p} \to e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$ *is an isomorphism of affine varieties.*

Proof. The adjoint action map is obviously a morphism of varieties, so we only need to show that it is bijective. Since $\mathfrak{g}_p$ has trivial center, we can identify $\mathfrak{g}_p$ with a subalgebra of $\operatorname{End} \mathfrak{g}_p$ through the map $\operatorname{ad} : \mathfrak{g}_p \to \operatorname{End} \mathfrak{g}_p$. Since ad is injective, we have $\mathfrak{n}_{\mathfrak{l},p} \cong \operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}$. The adjoint group of $\mathfrak{g}_p$ is the subgroup of $\operatorname{Aut}(\mathfrak{g}_p)$ generated by $\exp(\operatorname{ad} u)$ with u running through $\mathfrak{g}_p$, and $N_{\mathfrak{l},p}$ is the subgroup generated by $\exp(\operatorname{ad} v)$ with v running through $\mathfrak{n}_{\mathfrak{l},p}$. As $\mathfrak{n}_{\mathfrak{l},p}$ is nilpotent, the exponential map $\exp : \operatorname{ad} \mathfrak{n}_{\mathfrak{l},p} \to N_{\mathfrak{l},p}$ is surjective, i.e., every element of $N_{\mathfrak{l},p}$ can be expressed as $\exp(\operatorname{ad} v)$ for some $v \in \mathfrak{n}_{\mathfrak{l},p}$. Now we show that given an element $e + z \in e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$, there exists a unique element $e + y \in S_{e_p}$ and a unique element $x \in \mathfrak{n}_{\mathfrak{l},p}$, such that $\exp(\operatorname{ad} x)(e + y) = e + z$. Note that

$$\mathfrak{m}_{\mathfrak{l},p}^{\perp} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_p(i)\right) \oplus [\mathfrak{l}_p^{\perp}, e], \quad \mathfrak{n}_{\mathfrak{l},p} = \left(\bigoplus_{i \leq -2} \mathfrak{g}_p(i)\right) \oplus \mathfrak{l}_p^{\perp} \quad \text{and} \quad \mathfrak{g}_p^f \subseteq \bigoplus_{i \leq 0} \mathfrak{g}_p(i).$$

For an element $u \in \mathfrak{g}_p$, we write $u = \sum_i u_i$ with $u_i \in \mathfrak{g}_p(i)$. Let $x \in \mathfrak{n}_{\mathfrak{l},p}, y \in \mathfrak{g}_p^f$, and $z \in \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Then $x = \sum_{i \leq -1} x_i, y = \sum_{j \leq 0} y_j$ and $z = \sum_{k \leq 1} z_k$ with $x_{-1} \in \mathfrak{l}_p^{\perp}$ and $z_1 \in [\mathfrak{l}_p^{\perp}, e]$. Note that

$$\exp(\operatorname{ad} x)(e+y) = e + y + [x,e] + [x,y] + \sum_{n \geq 2} \frac{(\operatorname{ad} x)^n}{n!}(e+y).$$

The equation $\exp(\operatorname{ad} x)(e+y) = e + z$ means that

$$\sum_k z_k = \sum_j y_j + \sum_i [x_i, e] + \sum_{i,j} [x_i, y_j] + \sum_{n \geq 2} \frac{(\sum_i \operatorname{ad} x_i)^n}{n!}(e + \sum_j y_j), \tag{2.7}$$

which is equivalent to a series of equations, i.e., for $k \leq 1$,

$$\begin{aligned} z_k - y_k - [x_{k-2}, e] = \sum_{i+j=k} \operatorname{ad} x_i(y_j) + \sum_{n \geq 2} \frac{\sum_{i_1 + \cdots + i_n = k-2} \operatorname{ad} x_{i_1} \cdots \operatorname{ad} x_{i_n}(e)}{n!} \\ + \sum_{n \geq 2} \frac{\sum_{i_1 + \cdots + i_n + j = k} \operatorname{ad} x_{i_1} \cdots \operatorname{ad} x_{i_n}(y_j)}{n!}. \end{aligned} \tag{2.8}$$

We use a decreasing induction on k to show that given z, there is a unique solution (x, y) for (2.7). We remark that

- Given k, $\operatorname{ad} x_i, y_j$ appear on the right side of (2.8) only wehn $i > k-2$ and $j > k$. Moreover, if we have already found values for $\{x_i, y_j\}_{i \geq k_0 - 2, j \geq k_0}$ such that (2.8) is satisfied for all $k \geq k_0$, and if we only change the values of $\{x_i, y_j\}_{i < k_0 - 2, j < k_0}$, then (2.8) is still valid for $k \geq k_0$.
- We have the decomposition $\mathfrak{g}_p = \mathfrak{g}_p^f \oplus [\mathfrak{g}_p, e]$, i.e., $\mathfrak{g}_p(i) = \mathfrak{g}_p^f(i) \oplus [\mathfrak{g}_p(i-2), e]$, where $\mathfrak{g}_p^f(i) = \mathfrak{g}_p^f \cap \mathfrak{g}_p(i)$ for all i.
- $\operatorname{ad} e : \mathfrak{g}_p(i) \to \mathfrak{g}_p(i+2)$ is injective for $i \leq -1$.

When $k = 1$, (2.8) reads $[x_{-1}, e] = z_1$, which has a unique solution for x_{-1} when given z_1, as $x_{-1} \in \mathfrak{l}_p^{\perp}, z_1 \in [\mathfrak{l}_p^{\perp}, e]$ and $\operatorname{ad} e : \mathfrak{l}_p^{\perp} \to [\mathfrak{l}_p^{\perp}, e]$ is injective. For $k = k_0 \leq 0$, we assume that we have uniquely determined $\{x_i, y_j\}_{i \geq k_0 - 1, j \geq k_0 + 1}$ such that (2.8) is satisfied for $k \geq k_0 + 1$. We show that we can uniquely determine $(x_{k_0 - 2}, y_{k_0})$ (while $\{x_i, y_j\}_{i \geq k_0 - 1, j \geq k_0 + 1}$ will not change), such that (2.8) is satisfied for $k \geq k_0$. Set $k = k_0$ in (2.8), since the values of $\{x_i, y_j\}_{i \geq k_0 - 1, j \geq k_0 + 1}$ are already determined, the right side of (2.8) is determined, which is an element of $\mathfrak{g}_p(k_0)$. Denote it by w_{k_0}. Then (2.8) becomes $[x_{k_0 - 2}, e] = w_{k_0} + y_{k_0} - z_{k_0}$. This equation has a unique solution for (x_{k_0-2}, y_{k_0}) when z_{k_0} and w_{k_0} are given, as $\mathfrak{g}_p(k_0) = \mathfrak{g}_p^f(k_0) \oplus [\mathfrak{g}_p(k_0 - 2), e]$ and $\operatorname{ad} e$ is injective on $\mathfrak{g}_p(k_0 - 2)$. By induction, we can find a unique solution (x, y) for (2.7) when z is given. □

Remark 2.3.8. *The above isomorphism of affine varieties was proved in [Kos78] when e is a principal nilpotent element, and then generalized by W. Gan and V. Ginzburg in [GG02] for Dynkin good $\mathbb{Z}$-grading. Their proof involves a $\mathbb{C}^*$-action on both varieties and then applies a general theorem in algebraic geometry. Our proof here is purely algebraic and works for all good $\mathbb{Z}$-gradings.*

Corollary 2.3.9. The coadjoint action map $\alpha : N_{\mathfrak{l},p} \times \mathcal{S}_{\chi_p} \to \chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$ is an isomorphism of affine varieties, where $\mathfrak{m}_{\mathfrak{l},p}^{\perp,*} := \kappa_p(\mathfrak{m}_{\mathfrak{l},p}^{\perp})$.

2.3.3 Quantization of Slodowy slices

Recall the Kazhdan filtration on $U(\mathfrak{g}_p)$ induced by the $\mathbb{Z}$-grading Γ_p in Example 1.1.17. Let $\{U_n(\mathfrak{g}_p)\}$ be the PBW-filtration on $U(\mathfrak{g}_p)$ and

$$U_n(\mathfrak{g}_p)(i) := \{x \in U_n(\mathfrak{g}_p) \mid [h_\Gamma, x] = ix\}.$$

Then $K_nU(\mathfrak{g}_p) = \sum_{i+2j\leq n} U_j(\mathfrak{g}_p)(i)$. The Kazhdan filtration is separated and exhaustive, i.e.,

$$\bigcap_{n\in\mathbb{Z}} K_nU(\mathfrak{g}_p) = \{0\} \quad \text{and} \quad U(\mathfrak{g}_p) = \bigcup_{n\in\mathbb{Z}} K_nU(\mathfrak{g}_p).$$

The Kazhdan filtration on $U(\mathfrak{g}_p)$ induces filtrations on I_{χ_p}, Q_{χ_p} and H_{χ_p}, which we also denote by K_n. Moreover, $\mathrm{gr}_K I_{\chi_p}$ is just the ideal of $\mathbb{C}[\mathfrak{g}_p^*]$ defining the affine subvariety $\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, i.e., $\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$. Note that $K_nQ_{\chi_p} = 0$ for $n < 0$ as $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$ contains all the negative-degree generators of $U(\mathfrak{g}_p)$ with respect to the Kazhdan filtration.

Since $H_{\chi_p} \subseteq Q_{\chi_p}$, we have a natural inclusion map

$$\nu_1 : \mathrm{gr}_K H_{\chi_p} \to \mathrm{gr}_K Q_{\chi_p}.$$

On the other hand, as $\mathcal{S}_{\chi_p} \subseteq \chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, we have a restriction map

$$\nu_2 : \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}] \to \mathbb{C}[\mathcal{S}_{\chi_p}].$$

Composing these two maps, we get a homomorphism, as $\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$,

$$\nu = \nu_2 \circ \nu_1 : \mathrm{gr}_K H_{\chi_p} \to \mathbb{C}[\mathcal{S}_{\chi_p}].$$

We are going to show that ν is an isomorphism.

The module Q_{χ_p} is a filtered $U(\mathfrak{n}_{\mathfrak{l},p})$-module, where the filtration on $U(\mathfrak{n}_{\mathfrak{l},p})$ is the Kazhdan filtration induced from that of $U(\mathfrak{g}_p)$. This filtration induces filtrations on the cohomologies $H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})$, and there are canonical homomorphisms

$$h_i : \mathrm{gr}_K H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \to H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}). \tag{2.9}$$

Theorem 2.3.10. *The homomorphism $\nu : \mathrm{gr}_K H_{\chi_p} \to \mathbb{C}[\mathcal{S}_{\chi_p}]$ is an isomorphism.*

Proof. First, we show that $H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) = \delta_{i,0}\mathbb{C}[\mathcal{S}_{\chi_p}]$. Recall the isomorphism of affine varieties in Lemma 2.3.7, which is $N_{\mathfrak{l}_p}$-equivariant. Thus we have an $\mathfrak{n}_{\mathfrak{l},p}$-module isomorphism $\mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}] \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathbb{C}[\mathcal{S}_{\chi_p}]$. Hence

$$H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) = H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]) = H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}]) \otimes \mathbb{C}[\mathcal{S}_{\chi_p}].$$

The cohomology $H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}])$ is equal to the algebraic de Rham cohomology of $N_{\mathfrak{l}_p}$ [CE48], which is $\mathbb{C}$ for $i = 0$ and trivial for $i > 0$ as $N_{\mathfrak{l}_p}$ is isomorphic to an affine space.

Next we show that the homomorphisms h_i in (2.9) are all isomorphisms. The standard cochain complex for computing the cohomology of $\mathfrak{n}_{\mathfrak{l},p}$ with coefficients in Q_{χ_p} is

$$0 \to Q_{\chi_p} \to \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p} \to \cdots \to \Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p} \to \cdots . \tag{2.10}$$

Recall that there is a grading on $\mathfrak{g}_p^*$ hence a grading on $\mathfrak{n}_{\mathfrak{l},p}^*$, which is positively graded as $\mathfrak{n}_{\mathfrak{l},p}$ is negatively graded in $\mathfrak{g}_p$. We write the gradation as $\mathfrak{n}_{\mathfrak{l},p}^* = \bigoplus_{i\geq 1} \mathfrak{n}_{\mathfrak{l},p}^*(i)$. Define a filtration of $\Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p}$ by setting $F_s(\Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})$ to be the subspace spanned by $(x_1 \wedge \cdots \wedge x_n) \otimes v$ for all $x_i \in \mathfrak{n}_{\mathfrak{l},p}^*(n_i), v \in K_j Q_{\chi_p}$ such that $j + \sum n_i \leq s$, where K_j is the Kazhdan filtration on Q_{χ_p}. This defines a filtered complex on (2.10) whose associated graded complex gives us the standard cochain complex for computing the cohomology of $\mathfrak{n}_{\mathfrak{l},p}$ with coefficients in $\mathrm{gr}_K Q_{\chi_p}$.

Consider the spectral sequence with

$$E_0^{s,t} = \frac{F_s(\Lambda^{s+t}\mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})}{F_{s-1}(\Lambda^{s+t}\mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})}.$$

Then $E_1^{s,t} = H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, \dfrac{K_s Q_{\chi_p}}{K_{s-1} Q_{\chi_p}})$ and the spectral sequence converges to

$$E_\infty^{s,t} = \frac{F_s H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})}{F_{s-1} H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})},$$

i.e., the maps $h_i : \mathrm{gr}_K H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \to H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p})$ are isomorphisms hence

$$\mathrm{gr}_K H_{\chi_p} = \mathrm{gr}_K H^0(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \cong H^0(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) \cong \mathbb{C}[\mathcal{S}_{\chi_p}].$$

□

Remark 2.3.11. *For $p = 0$, the isomorphism in Theorem 2.3.10 was proved by A. Premet [Pre02] when $\mathfrak{l}$ is a Lagrangian subspace of $\mathfrak{g}(-1)$ and then generalized by W. Gan and V. Ginzburg [GG02] for general isotropic subspaces $\mathfrak{l}$. Our method here follows [GG02].*

Remark 2.3.12. *Theorem 2.3.10 shows that $(\mathbb{C}[\mathfrak{g}_p^*], \mathrm{gr}_K I_{\chi_p}, \mathbb{C}[\mathcal{S}_{\chi_p}])$ is a Poisson reducible triple and the Poisson structure on $\mathcal{S}_{\chi_p}$ can be considered as a Poisson reduction of $\mathfrak{g}_p^*$.*

Corollary 2.3.13. The algebra H_{χ_p} does not depend on the isotropic subspace $\mathfrak{l}_p$.

Proof. Let $\mathfrak{l}_p \subseteq \mathfrak{l}_p'$ be two isotropic subspaces of $\mathfrak{g}_p(-1)$, and H_{χ_p}, H_{χ_p}' the corresponding finite W-algebras. Then we have a natural map $\pi : H_{\chi_p} \hookrightarrow H_{\chi_p}'$ hence a natural map $\mathrm{gr}\,\pi : \mathrm{gr}_K H_{\chi_p} \hookrightarrow \mathrm{gr}_K H_{\chi_p}'$. By Theorem 2.3.10, we know that $\mathrm{gr}\,\pi$ is an isomorphism as they are both isomorphic to $\mathbb{C}[\mathcal{S}_{\chi_p}]$, so π is itself an isomorphism. □

Since H_{χ_p} does not depend on the isotropic subspace $\mathfrak{l}_p$, we choose it to be a Lagrangian subspace of $\mathfrak{g}_p(-1)$ from now on.

2.4 Kostant's theorem and Skryabin equivalence

2.4.1 Kostant's theorem

Given a finite-dimensional Lie algebra $\mathfrak{a}$ and a linear functional $\varphi \in \mathfrak{a}^*$, define

$$\mathfrak{a}^{\varphi} := \{x \in \mathfrak{a} \mid \varphi([x, y]) = 0 \text{ for all } y \in \mathfrak{a}\}.$$

The *index* of $\mathfrak{a}$ is defined to be $\chi(\mathfrak{a}) = \operatorname{Inf}\{\dim \mathfrak{a}^{\varphi} \mid \varphi \in \mathfrak{a}^*\}$. We say that $\varphi \in \mathfrak{a}^*$ is *regular* if $\dim \mathfrak{a}^{\varphi} = \chi(\mathfrak{a})$.

Given $x \in \mathfrak{a}$, let $\mathfrak{a}^x = \{y \in \mathfrak{a} \mid [x, y] = 0\}$ be the centralizer of x in $\mathfrak{a}$. Then x is called *regular* if its centralizer $\mathfrak{a}^x$ has minimal dimension, i.e., $\dim \mathfrak{a}^x \leq \dim \mathfrak{a}^{x'}$ for all $x' \in \mathfrak{a}$. When $\mathfrak{a}$ admits a non-degenerate invariant symmetric bilinear form which identifies $\mathfrak{a}$ and $\mathfrak{a}^*$, the regularity of an element is the same thing as the regularity of the corresponding linear function. It is well known that the subset of regular elements in $\mathfrak{a}$ is a dense open subset under the Zariski topology.

Let e be a regular nilpotent element in $\mathfrak{g}$, which we also call *principal nilpotent*. We show that the finite W-algebra H_{χ_p} associated to $(\mathfrak{g}_p, e)$ is isomorphic to $Z(\mathfrak{g}_p)$, the center of the universal enveloping algebra $U(\mathfrak{g}_p)$.

Let $S(\mathfrak{g}_p)$ be the symmetric algebra of $\mathfrak{g}_p$. It is well known that there is a canonical isomorphism of $\mathfrak{g}_p$-modules $\varphi : S(\mathfrak{g}_p) \to \mathrm{gr}U(\mathfrak{g}_p)$, where gr is the associated graded of the PBW filtration of $U(\mathfrak{g}_p)$. Let $I(\mathfrak{g}_p) := \{g \in S(\mathfrak{g}_p) \mid [x, g] = 0 \text{ for all } x \in \mathfrak{g}_p\}$ be the $\mathfrak{g}_p$-invariants in $S(\mathfrak{g}_p)$ and $Z(\mathfrak{g}_p)$ be the center of $U(\mathfrak{g}_p)$. Then the restriction of φ to $I(\mathfrak{g}_p)$ yields an isomorphism of vector spaces

$$\varphi : I(\mathfrak{g}_p) \to \mathrm{gr}Z(\mathfrak{g}_p).$$

Recall that $S_{e_p} = e + \mathfrak{g}_p^f$ and $\mathcal{S}_{\chi_p} = \kappa_p(S_{e_p})$. Since $\mathcal{S}_{\chi_p} \subseteq \mathfrak{g}_p^*$, we have a canonical restriction $\iota_p : \mathbb{C}[\mathfrak{g}_p^*] \to \mathbb{C}[\mathcal{S}_{\chi_p}]$. Identifying $\mathbb{C}[\mathfrak{g}_p^*]$ with $S(\mathfrak{g}_p)$ and restricting ι_p to $I(\mathfrak{g}_p)$, we get a natural map from $I(\mathfrak{g}_p)$ to $\mathbb{C}[\mathcal{S}_{\chi_p}]$, which we still denote by ι_p.

Lemma 2.4.1 ([RT92, MS16])**.** *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra and $x = \sum_i x_i t^i \in \mathfrak{g}_p$ with $x_i \in \mathfrak{g}$. Let e be a regular nilpotent element of $\mathfrak{g}$. Then*

(1) x is regular in $\mathfrak{g}_p$ if and only if x_0 is regular in $\mathfrak{g}$.

(2) Every element of S_{e_p} is regular. Moreover, the adjoint orbit of every regular element intersects S_{e_p} in a unique point.

(3) The map $\iota_p : I(\mathfrak{g}_p) \to \mathbb{C}[\mathcal{S}_{\chi_p}]$ is an isomorphism of vector spaces.

Theorem 2.4.2. *Let e be a regular nilpotent element of $\mathfrak{g}$. Then the finite W-algebra H_{χ_p} associated to the pair $(\mathfrak{g}_p, e)$ is isomorphic to the center of $U(\mathfrak{g}_p)$.*

Proof. Since $Z(\mathfrak{g}_p) \subseteq U(\mathfrak{g}_p)$ is obviously invariant under the adjoint action of $\mathfrak{n}_{\mathfrak{l},p}$, we have a natural map $j_p : Z(\mathfrak{g}_p) \to H_{\chi_p}$, which preserves the Kazhdan filtrations on $Z(\mathfrak{g}_p)$ and H_{χ_p}. Passing to their associated graded, we have $\mathrm{gr}\, j_p : \mathrm{gr}Z(\mathfrak{g}_p) \to \mathrm{gr}H_{\chi_p}$, which is the isomorphism $\iota : I(\mathfrak{g}_p) \to \mathbb{C}[\mathcal{S}_{\chi_p}]$. Since the associated graded of j_p is an isomorphism, j_p itself is an isomorphism of algebras.

$$\begin{array}{ccc} Z(\mathfrak{g}_p) & \xrightarrow{\ j_p\ } & H_{\chi_p} \\ \downarrow{\scriptstyle \mathrm{gr}} & & \downarrow{\scriptstyle \mathrm{gr}} \\ I(\mathfrak{g}_p) & \xrightarrow[\cong]{\ \mathrm{gr}\, j_p\ } & \mathbb{C}[\mathcal{S}_{\chi_p}] \end{array}$$

□

Remark 2.4.3. *When $p = 0$, i.e., in semi-simple cases, Lemma 2.4.1 and Theorem 2.4.2 were proved by B. Kostant [Kos78]. T. Macedo and A. Savage [MS16] generalized Lemma 2.4.1 to truncated multicurrent Lie algebras, on which non-degenerate invariant bilinear forms exist. Therefore, all the lemmas and theorems in this section can be generalized to those algebras, i.e., finite W-algebras associated to truncated multicurrent Lie algebras can be defined and Kostant's theorem holds.*

Remark 2.4.4. *Explicit generators of $I(\mathfrak{g}_p)$ were constructed in [RT92], but corresponding generators of $Z(\mathfrak{g}_p)$ are not known in general. When $\mathfrak{g} = s\ell_n$, A. Molev [Mol97] has given a description of generators of $Z(\mathfrak{g}_p)$.*

2.4.2 Skryabin equivalence

Definition 2.4.5. A $\mathfrak{g}_p$-module M is called a *Whittaker module* if $a - \chi_p(a)$ acts locally nilpotently on M for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. Given a Whittaker module M, an element $m \in M$ is called a *Whittaker vector* if $(a - \chi_p(a)) \cdot m = 0$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. Let $\mathrm{Wh}(M)$ be the collection of the Whittaker vectors of M.

Lemma 2.4.6. *The $\mathfrak{g}_p$-module Q_{χ_p} is a Whittaker module, with* $\mathrm{Wh}(Q_{\chi_p}) = H_{\chi_p}$.

Proof. Remember that $Q_{\chi_p} = U(\mathfrak{g}_p)/I_{\chi_p}$, where I_{χ_p} is the left ideal of $U(\mathfrak{g}_p)$ generated by $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. Since $\mathfrak{m}_{\mathfrak{l},p}$ is negatively graded in the good grading Γ_p of $\mathfrak{g}_p$, it acts nilpotently on $\mathfrak{g}_p$ hence locally nilpotently on $U(\mathfrak{g}_p)$. Note that $\mathrm{ad}\, a = \mathrm{ad}\,(a - \chi_p(a))$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$, so $\mathrm{ad}\,(a - \chi_p(a))$ acts locally nilpotently on $U(\mathfrak{g}_p)$, and also on its quotient Q_{χ_p}, i.e., Q_{χ_p} is a Whittaker module. Since we choose $\mathfrak{l}_p$ to be a Lagrangian subspace of $\mathfrak{g}_p(-1)$, we have $\mathfrak{n}_{\mathfrak{l},p} = \mathfrak{m}_{\mathfrak{l},p}$. Then by the definition of H_{χ_p}, we have $\mathrm{Wh}(Q_{\chi_p}) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p}) = H_{\chi_p}$. □

Let $\mathfrak{g}_p$-Wmod$^{\chi_p}$ be the category of finitely generated Whittaker $\mathfrak{g}_p$-modules and H_{χ_p}-Mod be the category of finitely generated left H_{χ_p}-modules.

Since $H_{\chi_p} \cong (\mathrm{End}_{\mathfrak{g}_p} Q_{\chi_p})^{op}$, Q_{χ_p} admits a right H_{χ_p}-module structure. Given $N \in H_{\chi_p}$-Mod, we have a $\mathfrak{g}_p$-module $Q_{\chi_p} \otimes_{H_{\chi_p}} N$ with $x \cdot (a \otimes n) := (x \cdot a) \otimes n$ for all $a \in Q_{\chi_p}, n \in N$.

Lemma 2.4.7. *(1) Let $M \in \mathfrak{g}_p$-Wmod$^{\chi_p}$. Then $\mathrm{Wh}(M) = 0$ implies that $M = 0$.*

(2) Let $M \in \mathfrak{g}_p$-Wmod$^{\chi_p}$. Then $\mathrm{Wh}(M)$ admits an H_{χ_p}-module structure, with $(y + I_{\chi_p}) \cdot v = y \cdot v$ for $y + I_{\chi_p} \in H_{\chi_p}, v \in M$.

(3) Let $N \in H_{\chi_p}$-Mod. Then $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p$-Wmod$^{\chi_p}$.

Proof. By definition, a Whittaker $\mathfrak{g}_p$-module M is locally $U(\mathfrak{m}_{\mathfrak{l},p})$-finite as $U(\mathfrak{m}_{\mathfrak{l},p})$ is generated by 1 and $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. Given a nonzero vector $v \in M$, we have $\dim U(\mathfrak{m}_{\mathfrak{l},p}) \cdot v < \infty$. Since $a - \chi_p(a)$ are nilpotent operators on $U(\mathfrak{m}_{\mathfrak{l},p}) \cdot v$, by Engel's theorem, we can find a nonzero common eigenvector for them, which is a Whittaker vector, so $\mathrm{Wh}(M) \neq 0$ if $M \neq 0$.

For (2), we only need to show that $y \cdot v \in \mathrm{Wh}(M)$ for all $y + I_{\chi_p} \in H_{\chi_p}$ and $v \in \mathrm{Wh}(M)$, because the module structure comes from the $U(\mathfrak{g}_p)$-module structure on M. We have

$$(a - \chi_p(a))y \cdot v = [a - \chi_p(a), y] \cdot v + y(a - \chi_p(a)) \cdot v = [a - \chi_p(a), y] \cdot v.$$

By the proof of Lemma 2.2.6, we have $[a, y] \in I_{\chi_p}$, so $(a - \chi_p(a))y \cdot v = 0$, i.e., $y \cdot v \in \mathrm{Wh}(M)$.

For (3), note that Q_{χ_p} is a Whittaker $\mathfrak{g}_p$-module, so $a - \chi_p(a)$ acts locally nilpotently on it. But the $U(\mathfrak{g}_p)$-action on the tensor product is from the left side, so $a - \chi_p(a)$ acts automatically locally nilpotently on the tensor product $Q_{\chi_p} \otimes_{H_{\chi_p}} N$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. □

By Lemma 2.4.7, we have two functors,

$$\begin{aligned} \mathrm{Wh}: \quad & \mathfrak{g}_p\text{-Wmod}^{\chi_p} \longrightarrow H_{\chi_p}\text{-Mod}, \qquad && M \longmapsto \mathrm{Wh}(M), \\ Q_{\chi_p} \otimes_{H_{\chi_p}} - : \quad & H_{\chi_p}\text{-Mod} \longrightarrow \mathfrak{g}_p\text{-Wmod}^{\chi_p}, \qquad && N \longmapsto Q_{\chi_p} \otimes_{H_{\chi_p}} N. \end{aligned}$$

The functor $\mathrm{Wh}(-)$ is left exact and the functor $Q_{\chi_p} \otimes_{H_{\chi_p}} -$ is right exact.

Theorem 2.4.8. *The two functors $\mathrm{Wh}(-)$ and $Q_{\chi_p} \otimes_{H_{\chi_p}} -$ give an equivalence of categories between $\mathfrak{g}_p$-Wmod$^{\chi_p}$ and H_{χ_p}-Mod.*

Proof. Since H_{χ_p} does not depend on the isotropic subspace $\mathfrak{l}_p$, we choose it to be a Lagrangian subspace of $\mathfrak{g}_p(-1)$, so we have $\mathfrak{m}_{\mathfrak{l},p} = \mathfrak{n}_{\mathfrak{l},p}$. First, we show that $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong N$ for all $N \in H_{\chi_p}$-Mod. Assume that N is generated by a finite-dimensional subspace N_0. Setting $K_n N := (K_n H_{\chi_p}) N_0$ gives a filtration on N and it becomes a filtered H_{χ_p}-module. We twist the $\mathfrak{m}_{\mathfrak{l},p}$-action on $Q_{\chi_p} \otimes_{H_{\chi_p}} N$ by $-\chi_p$, i.e., we define a new action by

$$a \cdot (u \otimes v) = (a - \chi_p(a))u \otimes v = \mathrm{ad}(a - \chi_p(a))(u) \otimes v \quad \text{ for } a \in \mathfrak{m}_{\mathfrak{l},p}, u \in Q_{\chi_p}, v \in N.$$

Then $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N)$ with respect to this new action. The Kazhdan filtrations on Q_{χ_p} and N induce a Kazhdan filtration on $Q_{\chi_p} \otimes_{H_{\chi_p}} N$, with

$$K_n(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = \sum_{i+j=n} K_i Q_{\chi_p} \otimes_{H_{\chi_p}} K_j N.$$

Since both $K_n Q_{\chi_p} = 0$ and $K_n N = 0$ for $n < 0$ as we noted in Section 2.3.3, the filtration gives us homomorphisms for $i \geq 0$,

$$h_i : \mathrm{gr}_K H^i(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \to H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N)). \tag{2.11}$$

Remember that $\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$ and $\mathrm{gr}_K H_{\chi_p} \cong \mathbb{C}[S_{\chi_p}] = \mathbb{C}[\chi_p + \ker \mathrm{ad}^* f]$. Since $\chi_p + \ker \mathrm{ad}^* f$ is an affine subspace of $\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, $\mathrm{gr}_K Q_{\chi_p}$ is free over $\mathrm{gr}_K H_{\chi_p}$, and we have an isomorphism

$$\mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong \mathrm{gr}_K Q_{\chi_p} \otimes_{\mathrm{gr}_K H_{\chi_p}} \mathrm{gr}_K N.$$

By Corollary 2.3.9, we have $\mathfrak{m}_{\mathfrak{l},p}$-module (precisely, $\mathfrak{n}_{\mathfrak{l},p}$-module) isomorphisms

$$\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathbb{C}[S_{\chi_p}] \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathrm{gr}_K H_{\chi_p}.$$

Therefore,

$$\begin{aligned} H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N)) &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p} \otimes_{\mathrm{gr}_K H_{\chi_p}} \mathrm{gr}_K N) \\ &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathrm{gr}_K N) \\ &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}]) \otimes \mathrm{gr}_K N \\ &= \delta_{i,0} \mathrm{gr}_K N. \end{aligned}$$

There is a spectral sequence as that in the proof of Theorem 2.3.10, which asserts that those h_i in (2.11) are all isomorphisms. Therefore, we have (note that $\mathrm{gr}_K N = N$)

$$H^i(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong \begin{cases} N & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases} \tag{2.12}$$

In particular, we have $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong N$.

Next we show that $Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \cong M$ for all $M \in \mathfrak{g}_p\text{-Wmod}^{\chi_p}$. Define a map

$$\varphi : Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \to M, \qquad (y + I_{\chi_p}) \otimes v \mapsto y \cdot v.$$

One can show that φ is a $\mathfrak{g}_p$-module homomorphism. Then we have the following exact sequence,

$$0 \to \ker \varphi \to Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \to M \to \mathrm{coker}\, \varphi \to 0. \tag{2.13}$$

Applying $\mathrm{Wh}(-)$ to the sequence (2.13), the identity $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = \mathrm{Wh}(M)$ and the left exactness of $\mathrm{Wh}(-)$ imply that $\mathrm{Wh}(\ker \varphi) = 0$, hence $\ker \varphi = 0$ by Lemma 2.4.7. Considering the long exact sequence of the cohomology of $\mathfrak{m}_{\mathfrak{l},p}$ associated to the sequence (2.13), we get

$$0 \to H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) \to H^0(\mathfrak{m}_{\mathfrak{l},p}, M) \to H^0(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{coker}\, \varphi) \to 0. \tag{2.14}$$

We stop at $H^0(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{coker}\varphi)$ because the next term $H^1(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = 0$ by (2.12). Note that $H^0(\mathfrak{m}_{\mathfrak{l},p}, -) = \mathrm{Wh}(-)$ and we already have $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = \mathrm{Wh}(M)$, so (2.14) implies that $\mathrm{Wh}(\mathrm{coker}\, \varphi) = 0$ hence $\mathrm{coker}\, \varphi = 0$, i.e., the map φ is an isomorphism. □

Remark 2.4.9. *Skryabin's original proof (see Appendix of [Pre02]) for Theorem 2.4.8 in the semi-simple case is different from our argument, which follows [GG02] and [Wan11].*

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