Boussinesq-like models high topographical variations

In this chapter, attention is paid to the regime of highly variable topographies $\beta = O(1)$. Like in the previous chapter, we aim at deriving asymptotic symmetric models, constructing approximate solutions of the water waves problem, and justifying this approximation. However, the previously exposed method cannot be applyied in the exact same way : this regime is indeed much more complex because of the greater influence of the topographical terms. These terms introduce new difficulties which compell us to revise and adapt our strategy. A new nonlinear change of variable is first performed and a single equivalent system is obtained. Adapting the first change of variable of the previous chapter, we are finally able to derive a class of equivalent systems via the introduction of four parameters this time. These ones can be chosen such that the systems are fully symmetric : they are proved to be well-posed and to provide a good approximation of the water waves solutions on a long time scale, under the assumption that the bottom is slowly variable.

3.1 A revised strategy

We recall the Boussinesq-like system $(\mathcal{B}_2)$ derived in Chapter 1, and the fact that solutions of the water waves problem are consistent with this system.

$$(\mathcal{B}_2)\begin{cases} \partial_t V + \nabla\eta + \frac{\varepsilon}{2}\nabla|V|^2 = 0 \ , \\ \partial_t\eta + \nabla\cdot(h\,V) + \varepsilon\Big[\nabla\cdot(\eta V) - \frac{1}{2}\nabla\cdot\Big(\frac{h^3}{3}\nabla\nabla\cdot V - h^2\nabla\nabla\cdot(h\,V)\Big)\Big] = 0 \ . \end{cases}$$

First remark that the bottom term h (recall that $h = 1 - b$ is the non-dimensional still water depth) appears in the first order term of the second equation of $(\mathcal{B}_2)$ whereas it is not present in the first one. This fact becomes important when it comes to the BBM trick which is unlikely to symetrize these terms. To correctly deal with this regime, we have to invert the order of the change of variables, and proceed with an adapted nonlinear change of variables first that symetrizes both order one terms and non-linear terms.

Taking into account the fact that we have to symmetrize both order one terms and nonlinear terms, we introduce the following change of variables :

$$\tilde{V} = \left(\sqrt{h} + \frac{\varepsilon}{2}\frac{\eta}{\sqrt{h}}\right) V \quad .$$

so that

$$V = \left(\frac{1}{\sqrt{h}} - \frac{\varepsilon}{2}\frac{\eta}{h\sqrt{h}}\right) \tilde{V} + O(\varepsilon^2) \quad .$$

Assuming that $\nabla\times\tilde{V} = O(\varepsilon)$, we formally derive the following system of equations satisfied by $\tilde{V}$ and η :

$$\begin{cases} \partial_t\tilde{V} + \sqrt{h}\nabla\eta + \dfrac{\varepsilon}{2\sqrt{h}}\Big[\dfrac{1}{2}\nabla\eta^2 + \dfrac{1}{2}\nabla|\tilde{V}|^2 + (\tilde{V}\cdot\nabla)\tilde{V} + \tilde{V}\nabla\cdot\tilde{V} \\ \qquad\qquad + \dfrac{1}{h}\Big(\dfrac{1}{2}(\nabla h\cdot\tilde{V})\tilde{V} - |\tilde{V}|^2\nabla h\Big)\Big] = O(\varepsilon^2)\ , \\ \partial_t\eta + \nabla(\sqrt{h}\cdot\tilde{V}) + \dfrac{\varepsilon}{2\sqrt{h}}\Big[\nabla\cdot(\eta\tilde{V}) - \sqrt{h}\nabla\cdot\Big(\dfrac{h^3}{3}\nabla\nabla\cdot(\dfrac{\tilde{V}}{\sqrt{h}}) \\ \qquad\qquad - h^2\nabla\nabla\cdot(\sqrt{h}\,\tilde{V})\Big) - \dfrac{\eta}{2h}\nabla h\cdot\tilde{V}\Big] = O(\varepsilon^2)\ . \end{cases}$$

We introduce the system (Γ_h) that corresponds to the homogeneous version of the previous system :

$$(S_h)\begin{cases} \partial_t V + \sqrt{h}\,\nabla\eta + \dfrac{\varepsilon}{2}F_h\begin{pmatrix} V \\ \eta \end{pmatrix} = 0\ , \\ \partial_t\eta + \nabla(\sqrt{h}\cdot V) + \dfrac{\varepsilon}{2}\Big[f_h\begin{pmatrix} V \\ \eta \end{pmatrix} - \nabla\cdot\Big(\dfrac{h^3}{3}\nabla\nabla\cdot(\dfrac{V}{\sqrt{h}}) - h^2\nabla\nabla\cdot(\sqrt{h}\,V)\Big)\Big] = 0\ , \end{cases}$$

where

$$\begin{cases} F_h\begin{pmatrix} V \\ \eta \end{pmatrix} &=& \dfrac{1}{\sqrt{h}}\Big(\dfrac{1}{2}\nabla\eta^2 + \dfrac{1}{2}\nabla|V|^2 + (V\cdot\nabla)V + V\nabla\cdot V \\ && + \dfrac{1}{h}\Big(\dfrac{1}{2}(\nabla h\cdot V)V - |V|^2\nabla h\Big)\Big)\ , \\ f_h\begin{pmatrix} V \\ \eta \end{pmatrix} &=& \dfrac{1}{\sqrt{h}}\Big(\nabla\cdot(\eta V) - \dfrac{\eta}{2h}\nabla h\cdot V\Big)\ . \end{cases}$$

On this new system (S_h), we have the following consistency result :

Proposition 3.1.1. *Consider a family* $(\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *of solutions of (0.0.7) such that* $(\nabla\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *is bounded with respect to* ε *in* $W^{1,\infty}([0, \frac{T}{\varepsilon}]; H^\sigma(\mathbb{R}^d)^{d+1})$ *with* σ *large enough. Then the family* $(V^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *is consistent with the system* (S_h)*, where* $V^\varepsilon = \nabla\psi^\varepsilon$*.*

Proof. First remark that since the velocity field V^ε is irrotationnal, we have $\nabla\times\tilde{V}^\varepsilon = O(\varepsilon)$. And since $(\nabla\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is consistent with the Boussinesq-like system $(\mathcal{B}_2)$, the previous computations yield directly the result. □

3.2 A new class of equivalent systems

In the previous chapter, we saw that a suitable change of variable comes from considering V_θ, the horizontal component of the velocity at the height $-1+\theta(1+\varepsilon(\eta-b)$ with $\theta \in [0,1]$, instead of the horizontal component of the velocity field at the free surface. We can remark that the link between these two variables (and hence the adequate change of variables) can be derived from the expression of u_{app} computed during the asymptotic expansion process of the operator $Z_\varepsilon(\varepsilon\eta, \beta b)$, which implies that we must adapt the change of variable for the strong variations regime since the expression of u_{app} now strongly depends on the topography.

We saw in the Chapter 1 that the computation of the asymptotic expansion of $Z_\varepsilon(\eta, b)\psi$ relies on finding an approximate solution of the elliptic problem (H) on the band $\mathcal{S} = [-1,0]\times\mathbb{R}^2$. Starting from the truncation of the computed value of u_{app} at the order $O(\varepsilon^2)$,

$$u_{app} = \psi + \varepsilon\left[\left(\frac{1}{2} - \frac{(z+1)^2}{2}\right) h^2\,\Delta\psi - zh\nabla h\cdot\nabla\psi\right] + O(\varepsilon^2)\ ,$$

where ψ is the value of the velocity potential at the free surface, shows that $\nabla u_{app}(\cdot, z)$ gives an approximation at order ε^2 of the horizontal component of the velocity field, namely $V(\cdot, z) = \nabla\phi(\cdot, z)$ at height $z \in [-1, 0]$.
Consequently, in presence of large bottom variations, the adequate change of variables is given by :

$$V_\theta = \left[1 - \frac{\varepsilon}{2}(\theta-1)(\theta\nabla(h^2\nabla\cdot\,) + \nabla\nabla\cdot(h^2\,.\,))\right]\,V\ ,$$

so that

$$V = \left[1 + \frac{\varepsilon}{2}(\theta-1)(\theta\nabla(h^2\nabla\cdot\,) + \nabla\nabla\cdot(h^2\,.\,))\right]\,V_\theta + O(\varepsilon^2)\ .$$

From this change of variables, we easily compute the expressions of $\partial_t V$ and $\nabla\cdot\sqrt{h}V$ which we plug into the system (S_h). By rewriting carefully the topography terms in order to make the quantity $\sqrt{h}$ appear, one gets the following system :

$$\begin{cases} \partial_t V_\theta + \sqrt{h}\,\nabla\eta + \frac{\varepsilon}{2}\Big[F_h\begin{pmatrix} V_\theta \\ \eta\end{pmatrix} + \nabla\Big((\theta^2-1)h^2\nabla\cdot\partial_t V_\theta + 2(\theta-1)h\nabla h\cdot\partial_t V_\theta\Big)\Big] \\ \qquad\qquad = O(\varepsilon^2)\ , \\ \partial_t\eta + \nabla(\sqrt{h}\cdot V_\theta) + \frac{\varepsilon}{2}\Big[f_h\begin{pmatrix} V_\theta \\ \eta\end{pmatrix} - \nabla\cdot\Big((\theta^2-\frac{1}{3})h^2\nabla\nabla\cdot(\sqrt{h}V_\theta) \\ \qquad +(\frac{3}{2}\theta^2-\frac{7}{6})h\nabla h\nabla\cdot(\sqrt{h}V_\theta) - \frac{(\theta-2)^2}{2}\sqrt{h}\nabla h(\nabla h\cdot V_\theta) \\ \qquad -(\frac{\theta^2}{2}-2\theta+\frac{7}{6})h\sqrt{h}\nabla(\nabla h\cdot V_\theta)\Big)\Big] = O(\varepsilon^2)\ . \end{cases}$$

At this point a new problem arises. Applying the BBM trick in the exact same way as in the previous regime leads to a system that is never symmetric for any values of the parameters θ, λ_1 and λ_2. Indeed, this requires the resolution of an over-determined numerical system on the unknowns θ, λ_1 and λ_2. To handle this problem, we simply introduce an additionnal unknown during the BBM trick process :

$$\begin{cases} \partial_t V_\theta = (1-\lambda_1)\partial_t V_\theta - \lambda_1\sqrt{h}\nabla\eta + O(\varepsilon)\ \ , \\ \partial_t V_\theta = (1-\lambda_2)\partial_t V_\theta - \lambda_2\sqrt{h}\nabla\eta + O(\varepsilon)\ \ , \\ \nabla\cdot(\sqrt{h}V_\theta) = \lambda_3\nabla\cdot(\sqrt{h}V_\theta) - (1-\lambda_3)\partial_t\eta + O(\varepsilon)\ \ , \end{cases}$$

and we use the first relation on the term $(\theta^2-1)h^2\nabla\cdot\partial_t V_\theta$ and the second relation on the term $2(\theta-1)h\nabla h\cdot\partial_t V_\theta$.

Finally, we formally derive a new class $\mathcal{T}^h$ of systems, and we easily prove that if a family $(V^\varepsilon,\eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is consistent with the system (S_h) then $(V_\theta^\varepsilon,\eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ - where $V_\theta^\varepsilon = \Big[1-\frac{\varepsilon}{2}(\theta-1)(\theta\nabla(h^2\nabla\cdot\ ) + \nabla\nabla\cdot(h^2\ .\ ))\Big]V^\varepsilon$ - is consistent with any of the following systems $(T_{\theta,\lambda_1,\lambda_2,\lambda_3}^{\ \ b})$:

$$(T_{\theta,\lambda_1,\lambda_2,\lambda_3}^{\ \ b})\begin{cases} \Big(1-\frac{\varepsilon}{2}\mathcal{P}_h^1\Big)\partial_t V + \sqrt{h}\,\nabla\eta + \frac{\varepsilon}{2}\Big[F_h\begin{pmatrix} V \\ \eta\end{pmatrix} + b_1\sqrt{h}\nabla\nabla\cdot(h^2\nabla\eta) \\ +b_2\sqrt{h}\nabla(h\nabla h\cdot\nabla\eta) + b_3\nabla h\nabla\cdot(h\sqrt{h}\nabla\eta) + b_4\sqrt{h}\nabla h(\nabla h\cdot\nabla\eta)\Big] = 0\ , \\ \Big(1-\frac{\varepsilon}{2}\mathcal{P}_h^2\Big)\partial_t\eta + \nabla(\sqrt{h}\cdot V) + \frac{\varepsilon}{2}\Big[f_h\begin{pmatrix} V \\ \eta\end{pmatrix} + \nabla\cdot\Big(c_1h^2\nabla\nabla\cdot(\sqrt{h}V) \\ +c_2h\nabla h\nabla\cdot(\sqrt{h}V) + c_3h\sqrt{h}\nabla(\nabla h\cdot V) + c_4\sqrt{h}\nabla h(\nabla h\cdot V)\Big)\Big] = 0\ . \end{cases}$$

where the operators $\mathcal{P}_h^1$ and $\mathcal{P}_h^2$ are defined by

$$\begin{cases} \mathcal{P}_h^1 = (1-\theta)\Big((1-\lambda_1)(\theta+1)\nabla(h^2\nabla\cdot\quad) + 2(1-\lambda_2)\nabla(h\nabla h\cdot\quad)\Big)\ \ , \\ \mathcal{P}_h^2 = (1-\lambda_3)\Big((\theta^2-\frac{1}{3})\nabla\cdot(h^2\nabla\quad) + (\frac{3}{2}\theta^2-\frac{7}{6})\nabla\cdot(h\nabla h\times\quad)\Big)\ \ , \end{cases}$$

and the parameters $(a_i)_{1\leq i\leq 4}, (b_i)_{1\leq i\leq 4}$ have the following expressions :

$$\begin{cases} b_1 = \lambda_1(1-\theta^2); & c_1 = \lambda_3(\theta^2 - \frac{1}{3}); \\ b_2 = (1-\theta)(2\lambda_2 - \frac{3}{2}\lambda_1(1+\theta)); & c_2 = \lambda_3(\frac{3}{2}\theta^2 - \frac{7}{6}); \\ b_3 = \frac{\lambda_1}{2}(1-\theta^2); & c_3 = -\frac{1}{2}\theta^2 + 2\theta - \frac{7}{6}; \\ b_4 = (1-\theta)(\lambda_2 - \frac{\lambda_1}{2}(1+\theta)); & c_4 = \frac{1}{2}(\theta-2)^2; \end{cases}$$

The previous computations are summed up in the following proposition.

Proposition 3.2.1. *Let $\theta \in [0,1]$ and $(\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ a family of solutions of (0.0.7) such that $(\nabla\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded with respect to ε in $W^{1,\infty}([0, \frac{T}{\varepsilon}]; H^\sigma(\mathbb{R}^d)^{d+1})$ with σ large enough. We define $V^\varepsilon = \nabla\psi^\varepsilon$ and*

$$\tilde{V}^\varepsilon = \left(1 - \frac{\varepsilon}{2}(\theta-1)(\theta\nabla(h^2\nabla\cdot\ ) + \nabla\nabla\cdot(h^2\ .\ ))\right)\left(\sqrt{h} + \frac{\varepsilon}{2}\frac{\eta}{\sqrt{h}}\right) V^\varepsilon.$$

Then for all $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, the family $(\tilde{V}^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is consistent with the system $(T^{\ b}_{\theta,\lambda_1,\lambda_2,\lambda_3})$.

Moreover, we have the following proposition on the existence of a subclass of $\mathcal{T}^h$ composed with fully symmetric systems.

Proposition 3.2.2. *There exists at least one value of $(\theta, \lambda_1, \lambda_2, \lambda_3)$ such that the system $(T^{\ b}_{\theta,\lambda_1,\lambda_2,\lambda_3})$ is fully symmetric.*

Proof. We are concerned here with the resolution of the following system :

$$\begin{cases} b_1 = c_1 \ , \\ b_2 = -c_2 \ , \\ b_3 = c_3 \ , \\ b_4 = -c_4 \ . \end{cases}$$

This system on $(\theta, \lambda_1, \lambda_2, \lambda_3)$ have at least one solution that gives the following approximate values :

$$\begin{cases} \theta \approx 0.6318 \ , \\ \lambda_1 \approx -0.3416 \ , \\ \lambda_2 \approx -2.8209 \ , \\ \lambda_3 \approx -3.1157 \ , \end{cases}$$

which ends the proof. □

From now on, we only consider this solution and its approximate values.

3.3 The final class of symmetric models

Thanks to Proposition 3.2.2, we know that some of the systems $(T_{\theta,\lambda_1,\lambda_2,\lambda_3}^{\ \ \ b})$ of the class $\mathcal{T}^h$ are completely symmetric : we hence denote by Σ^h the non-empty subclass of $\mathcal{S}^b$ composed with these symmetric systems. Unfortunately, we do not have the same kind of existence theory on these systems as in the previous regime. Indeed, the main difference consists in the order one terms of the two equations $\begin{pmatrix} \sqrt{h}\nabla\eta \\ \nabla\cdot(\sqrt{h}V) \end{pmatrix}$. In order to focus on the problem, we rewrite these terms $A(X,\partial_X)\begin{pmatrix} V \\ \eta \end{pmatrix}$ where $A(X,\partial_X) = \begin{pmatrix} 0 & \sqrt{h}\nabla_X \\ \nabla_X\cdot(\sqrt{h}\times\ ) & 0 \end{pmatrix}$. The proof of the existence of solutions on a short time scale is not modified by these terms, the classical proof is still valid. However, the fact that the matrix A depends on the bottom term h is a real problem as far as the large time existence is concerned : indeed, one crucial point of the proof here relies on the size of the quantity $\dfrac{\nabla h}{\varepsilon}$ on which we have no piece of information. The only case wherein we are surely able to demonstrate the large time existence is the case where ∇h is of order $O(\varepsilon)$: the term $\dfrac{\nabla h}{\varepsilon}$ is then of order $O(1)$ and we can conclude. In all other cases, the classical proof fails to provide a rigourous demonstration of the long time existence of solutions to these symmetric systems. Nevertheless, we are able to state the following proposition :

Proposition 3.3.1. *Let $s > \dfrac{d}{2}+1$ and $(\theta,\lambda_1,\lambda_2,\lambda_3)$ be such that the system $(T_{\theta,\lambda_1,\lambda_2,\lambda_3}^{\ \ \ b})$ belongs to the class Σ^h.*
Then for all $(V_0,\eta_0)\in H^s(\mathbb{R}^d)^{d+1}$, there exists a time T_0 independant of ε and a unique solution $(V,\eta)\in C([0,T_0];H^s(\mathbb{R}^d)^{d+1})\cap C^1([0,T_0];H^{s-3}(\mathbb{R}^d)^{d+1})$ to the system $(T_{\theta,\lambda_1,\lambda_2,\lambda_3}^{\ \ \ b})$ such that $(V,\eta)_{|t=0}=(V_0,\eta_0)$.
Furthermore, this unique solution is bounded independently of ε in the following sense : there exists a constant C_0 independent of ε such that for all k verifying $s-3k>\dfrac{d}{2}+1$, we have :

$$|(V,\eta)|_{W^{k,\infty}([0,T_0];H^{s-3k}(\mathbb{R}^d)^{d+1})}\leq C_0\ \ .$$

Besides, if we suppose that $\nabla b=O(\varepsilon)$, the previous result becomes valid on the long time interval $[0,\dfrac{T_0}{\varepsilon}]$.

Proof. The key point of the proof is to demonstrate that the elliptic operator $1-\dfrac{\varepsilon}{2}\begin{pmatrix} \mathcal{P}_h^1 \\ \mathcal{P}_h^2 \end{pmatrix}$ is a positive one. We first focus on $\mathcal{P}_h^1$:

$$(1-\frac{\varepsilon}{2}\mathcal{P}_h^1 V,V) = |V|_2^2+\frac{\varepsilon}{2}(1-\theta^2)(1-\lambda_1)|h\nabla\cdot V|_2^2+\varepsilon(1-\theta)(1-\lambda_2)(\nabla h\cdot V,h\nabla\cdot V)$$

Using the following inequality (satisfied for all $a\in\mathbb{R}$) :

$$\Big|(\nabla h\cdot V,h\nabla\cdot V)\Big|\leq\frac{a^2}{2}|h\nabla\cdot V|_2^2+\frac{1}{2a^2}|\nabla h\cdot V|_2^2\ \ ,$$

and taking $a^2 = \dfrac{(1+\theta)(1-\lambda_1)}{1-\lambda_2}$ leads to :

$$(1-\frac{\varepsilon}{2}\mathcal{P}_h^1 V, V) \geq |V|_2^2 - \frac{\varepsilon}{2}\frac{(1-\theta)(1-\lambda_2)^2}{(1+\theta)(1-\lambda_1)}|\nabla h\cdot V|_2^2$$

Using the classical Cauchy-Schwarz inequality leads finally to :

$$(1-\frac{\varepsilon}{2}\mathcal{P}_h^1 V, V) \geq \Big(1-\frac{\varepsilon}{2}\frac{(1-\theta)(1-\lambda_2)^2}{(1+\theta)(1-\lambda_1)}|\nabla h|_2^2\Big)\,|V|_2^2\ ,$$

At this point, if we take a small enough value of ε, f.e. $\varepsilon \leq \dfrac{2(1+\theta)(1-\lambda_1)}{(1-\theta)(1-\lambda_2)^2\,|\nabla h|_2^2}$, it ensures the global positivity of $\mathcal{P}_h^1$. On $\mathcal{P}_h^2$, we use the same method :

$$(1-\frac{\varepsilon}{2}\mathcal{P}_h^2\eta,\eta) = |\eta|_2^2 + \frac{\varepsilon}{2}(1-\lambda_3)(\theta^2-\frac{1}{3})|h\nabla\eta|_2^2 + \frac{\varepsilon}{2}(1-\lambda_3)(\frac{3}{2}\theta^2-\frac{7}{6})(\eta\nabla h, h\nabla\eta)$$

Using the same ideas as previously, one gets :

$$(1-\frac{\varepsilon}{2}\mathcal{P}_h^2\eta,\eta) \geq \Big(1-\frac{\varepsilon}{8}\frac{(1-\lambda_3)(\frac{3}{2}\theta^2-\frac{7}{6})^2}{\theta^2-\frac{1}{3}}\,|\nabla h|_2^2\Big)\,|\eta|_2^2\ ,$$

Once more, if we take f.e. $\varepsilon \leq \dfrac{8(\theta^2-\frac{1}{3})}{(1-\lambda_3)(\frac{3}{2}\theta^2-\frac{7}{6})^2\,|\nabla h|_2^2}$, we have the global positivity of $\mathcal{P}_h^2$.

Consequently, taking $\varepsilon \leq \min(\dfrac{2(1+\theta)(1-\lambda_1)}{(1-\theta)(1-\lambda_2)^2\,|\nabla h|_2^2}, \dfrac{8(\theta^2-\frac{1}{3})}{(1-\lambda_3)(\frac{3}{2}\theta^2-\frac{7}{6})^2\,|\nabla h|_2^2})$ ensures that the operator $1-\dfrac{\varepsilon}{2}\begin{pmatrix}\mathcal{P}_h^1\\ \mathcal{P}_h^2\end{pmatrix}$ is positive.

At this point, using this result and performing usual energy estimates on the system proves the existence of a time T such that there exists an unique solution $(V,\eta)\in C([0,T];H^s(\mathbb{R}^d)^{d+1})\cap C^1([0,T];H^{s-3}(\mathbb{R}^d)^{d+1})$ to the system. □

This result gives an efficient theoretical background to contruct approximate solutions of the water waves problem on a time scale $O(1)$, and $O(\frac{1}{\varepsilon})$ in the case $\nabla h = O(\varepsilon)$.

This contruction follows the same steps - but in a different order - as the contruction of approximate solutions for the first regime : we consider a solution $(\psi^\varepsilon,\eta^\varepsilon)$ to the formulation (0.0.7) of the water waves problem. We take initial data $(\psi_0^\varepsilon,\eta_0^\varepsilon)$ such that $(\nabla\psi_0^\varepsilon,\eta_0^\varepsilon)\in H^s(\mathbb{R}^d)^{d+1}$ for a suitably large value of s. We then define $V^\varepsilon = \nabla\psi^\varepsilon$ and $V_0^\varepsilon = \nabla\psi_0^\varepsilon$: we first contruct the data $(V_{\Sigma,0}^\varepsilon,\eta_{\Sigma,0}^\varepsilon)$ by applying the two successive changes of variable on the data $(V_0^\varepsilon,\eta_0^\varepsilon)$. We then choose the parameters $(\theta,\lambda_1,\lambda_2,\lambda_3)\in[0,1]\times\mathbb{R}^3$ such that the system $(T^b_{\theta,\lambda_1,\lambda_2,\lambda_3})$ is completely symmetric. Using Proposition 3.3.1, we know that there exists a unique solution to this system with initial data $(V_{\Sigma,0}^\varepsilon,\eta_{\Sigma,0}^\varepsilon)$: we denote this solution by $(V_\Sigma^\varepsilon,\eta_\Sigma^\varepsilon)$. From this exact solution of the symmetric system

($T^{\,b}_{\theta,\lambda_1,\lambda_2,\lambda_3}$), we finally construct an approximate solution of the water waves problem by successively and approximatively inverting the two changes of variable as shown below (which is possible if ε is small enough) :

$$\left\{\begin{array}{lcl} V^\varepsilon_{app} & = & \left(\dfrac{1}{\sqrt{h}} - \dfrac{\varepsilon}{2}\dfrac{\eta^\varepsilon}{h\sqrt{h}}\right)\left(1+\dfrac{\varepsilon}{2}(\theta-1)(\theta\nabla(h^2\nabla\cdot V^\varepsilon_\Sigma)+\nabla\nabla\cdot(h^2\,V^\varepsilon_\Sigma))\right) \\ \eta^\varepsilon_{app} & = & \eta^\varepsilon_\Sigma \end{array}\right.$$

We are now able to state our final result :

Theorem 3.3.2. *Let $T_1 \geq 0$, $s \geq \dfrac{d}{2}+1$, $\sigma \geq s+3$ and $(\nabla\psi^\varepsilon_0, \eta^\varepsilon_0)$ be in $H^\sigma(\mathbb{R}^d)^{d+1}$. Let $(\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a family of solutions of (0.0.7) with initial data $(\psi^\varepsilon_0, \eta^\varepsilon_0)_{0<\varepsilon<\varepsilon_0}$ such that $(\nabla\psi^\varepsilon, \eta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded in $W^{1,\infty}([0,T_1]; H^\sigma(\mathbb{R}^d)^{d+1})$. We define $V^\varepsilon = \nabla\psi^\varepsilon$ and choose $(\theta,\lambda_1,\lambda_2,\lambda_3) \in [0,1]\times\mathbb{R}^3$ such that the system $(T^{\,b}_{\theta,\lambda_1,\lambda_2,\lambda_3}) \in \Sigma^h$.*
Then for all $\varepsilon < \varepsilon_0$, there exists a time $T \leq T_1$ such that we have :

$$|V^\varepsilon - V^\varepsilon_{app}|_{L^\infty([0,T];H^s)} + |\eta^\varepsilon - \eta^\varepsilon_{app}|_{L^\infty([0,T];H^s)} \leq C\,\varepsilon^2$$

Besides, if we suppose that $\nabla h = O(\varepsilon)$ then $(V^\varepsilon_{app}, \eta^\varepsilon_{app})$ approximates the water waves solutions on a large time scale :

$$\forall t \in [0, \frac{T}{\varepsilon}]\ , \quad |V^\varepsilon - V^\varepsilon_{app}|_{L^\infty([0,t];H^s)} + |\eta^\varepsilon - \eta^\varepsilon_{app}|_{L^\infty([0,t];H^s)} \leq C\,\varepsilon^2 t$$

Proof. The proof is an adaptation of the one of Theorem 2.3.2, and we omit it here. □

Remark 3.3.3. *In the general case, where we have no piece of information on the size of the quantity $\dfrac{\nabla h}{\varepsilon}$, our analysis is only complete on a short time scale. We have indeed an approximation on this interval of time, and we know from Lannes [41] the existence of solutions to the water waves problem on a short time scale in 1-D and 2-D surface. However, if we suppose that $\nabla h = O(\varepsilon)$, this analysis is totally complete - like in the first regime - since we know from Alvarez-Samaniego and Lannes [3] the existence of solutions to the water waves problem on a long time scale in 1-D and 2-D surface.*

Remark 3.3.4. *The regime of long wave ($\varepsilon = \mu \ll 1$ where $\mu = \dfrac{h^2}{\lambda^2}$) and strong topography variations ($\beta = O(1)$) considered here can be seen as a particular case of the Green-Naghdi regime ($\mu \ll 1$ and no particular assumption on ε and β) derived in [27] and fully justified in [3, 4].*

PARTIE II

Sur l'approximation de Korteweg-de Vries pour fond non plat