A relaxation method for two-phase flow models with hydrodynamic closure law

Michaël BAUDIN[1], Christophe BERTHON[2], Frédéric COQUEL[3], Roland MASSON[1] and Quang Huy TRAN[1]

[1] IFP, 1 et 4 avenue de Bois-Préau, 92852 Rueil-Malmaison Cedex, France
[2] MAB, Université de Bordeaux I, 351 cours de la Libération, 33405 Talence Cedex, France
[3] Lab. J.-L. Lions, Université Pierre et Marie Curie, Boîte courrier 187, 75252 Paris Cedex 5

Abstract The present paper is devoted to the numerical approximation of the solutions of a system of conservation laws arising in the modeling of two-phase flows in oil and gas pipeline. The PDEs are closed by two (generally speaking) highly nonlinear algebraic relations : namely a pressure law and a hydrodynamic one. The severe nonlinearities they encode make the classical approximate Riemann solvers to be virtually intractable at a reasonable cost of evaluation. We propose an original strategy for relaxing solely these two nonlinearities : the relaxation system we introduce is nonlinear and is analyzed on the ground of a Chapman-Enskog like expansion. Our model is of course hyperbolic but all the associated fields are linearly degenerate. Such a property not only makes trivial the resolution of the Riemann problem but also unables us to enforce for validity several stability requirements. The new method turns out to be fairly simple and robust while achieving desirable positivity properties on the density and mass fractions. Numerical evidences are proposed.

Key words Two-phase flows, finite volume schemes, relaxation

Introduction

The petroleum pipelines drive a mixing, made up essentially of oil, gas and water, over a long distance. Locally, variations in the topography strongly influence the nature of the mixing : stratified, emulsion, slug for instance (see [26], [50] and the references therein). Several models must be then considered, each of them corresponding to a specific flow regime. All of the models we shall be considering are characterized by the choice of two algebraic closures : the first one defines the pressure of a mixing gas-liquid as a function of its composition ; the second one prescribes the hydrodynamic law, e.g., the velocity difference between the two phases of the flow as a function of the unknowns, such a function being defined according to the local incline of the pipeline.

The model considered in this paper describes a two-phase flow inside a pipeline with a uniform section but for general pressure and hydrodynamic laws. The main interest of dealing with such a model is to focuse attention on the difficulties associated with the hydrodynamic law, which besides the pressure law, gives rise to the main nonlinearities in the model.

In the flow, the gas (resp. liquid) is characterized by its density $\rho_{\rm G}$ (resp. $\rho_{\rm L}$), its velocity $v_{\rm G}$ (resp. $v_{\rm L}$) and its surface fraction $R_{\rm G} \in [0,1]$ (resp. $R_{\rm L} \in [0,1]$) with the property $R_{\rm L} + R_{\rm G} = 1$. The model is governed by the following system of conservation laws :

$$\left\{ \begin{array}{lll} \partial_t \left(\rho_{\rm L} R_{\rm L} \right) & + \partial_x \left(\rho_{\rm L} R_{\rm L} v_{\rm L} \right) & = 0, \\ \partial_t \left(\rho_{\rm G} R_{\rm G} \right) & + \partial_x \left(\rho_{\rm G} R_{\rm G} v_{\rm G} \right) & = 0, \\ \partial_t \left(\rho_{\rm L} R_{\rm L} v_{\rm L} + \rho_{\rm G} R_{\rm G} v_{\rm G} \right) & + \partial_x \left(\rho_{\rm L} R_{\rm G} v_{\rm L}^2 + \rho_{\rm G} R_{\rm G} v_{\rm G}^2 + p \right) & = 0. \end{array} \right. \tag{3.1}$$

for all $x \in \mathbb{R}$ and $t > 0$ where the unknown is $\mathbf{w} := (\rho_{\rm L} R_{\rm L},\ \rho_{\rm G} R_{\rm G},\ \rho_{\rm L} R_{\rm L} v_{\rm L} + \rho_{\rm G} R_{\rm G} v_{\rm G})$. In addition to the pressure law $p = p(\mathbf{w}_1,\ \mathbf{w}_2)$, assumed to be a given smooth function, we consider a general algebraic hydrodynamic law [54] of the type

$$v_{\rm L} - v_{\rm G} = \Phi(\mathbf{w}), \quad \text{for all} \ \ \mathbf{w}_1 \geq 0,\ \mathbf{w}_2 \geq 0,\ \mathbf{w}_3 \in \mathbb{R}, \tag{3.2}$$

in order to close (3.1). In (3.2), the mapping Φ is assumed to be smooth enough and in practical situations, turns out to be nonlinear in the unknown $\mathbf{w}$ (see [11], [54], [71] for instance). Consequently, when working with (3.1)–(3.2), the velocities $v_{\rm L}$ and $v_{\rm G}$ of the two phases must be understood as nonlinear functions of the conservative unknown $\mathbf{w}$:

$$\begin{aligned} v_{\rm L}(\mathbf{w}) &:= \frac{\mathbf{w}_3}{\mathbf{w}_1 + \mathbf{w}_2} + \frac{\mathbf{w}_2}{\mathbf{w}_1 + \mathbf{w}_2}\ \Phi(\mathbf{w}), \\ v_{\rm G}(\mathbf{w}) &:= \frac{\mathbf{w}_3}{\mathbf{w}_1 + \mathbf{w}_2} - \frac{\mathbf{w}_1}{\mathbf{w}_1 + \mathbf{w}_2}\ \Phi(\mathbf{w}). \end{aligned} \tag{3.3}$$

These formulae clearly highlight that the flux function is in full generality highly nonlinear !

Besides this observation, such nonlinearities are obviously responsible for the basic mathematical properties of the model. For instance, considering the simplest framework : namely a pressure law satisfying the assumption $\mathbf{w}_1 \frac{\partial p}{\partial \mathbf{w}_1} + \mathbf{w}_2 \frac{\partial p}{\partial \mathbf{w}_2} > 0$ together with a no-slip hydrodynamic law, i.e., $\Phi \equiv 0$, then the system (3.1) can be shown to be hyperbolic (see for instance Benzoni-Gavage [11]). However, as soon as $\Phi \not\equiv 0$, the system (3.1) is generally only conditionally hyperbolic : indeed for $|v_{\rm L} - v_{\rm G}| = |\Phi|$ large enough the hyperbolicity property is lost. To cap it all, arbitrary hydrodynamic closure (3.2) precludes the existence of additional non trivial conservation law for smooth solution of (3.1)–(3.2). In other words and without restrictive physical assumptions (like $\Phi = 0$), the system under consideration cannot be endowed with an entropy pair.

Turning considering the issue of approximating the solutions of (3.1)-(3.2), let us stress out that the algebraic complexity due to the above mentionned nonlinearities prevents us from using classical approximate Riemann solvers, whenever a low cost of computational effort is a critical issue. Let us simply recall that the latters strongly make use of the eigen-structure of the exact Jacobian matrix (see [30], [33], [35], [56] ...). Notice that even for simpler methods, like the well-known Lax-Friedrichs scheme (see Godlewski-Raviart [33]), one needs an estimate of the largest eigenvalue of the Jacobian matrix. In the present work, we propose a strategy of relaxation of the two main nonlinearities involved in (3.1)-(3.2). Since no entropy pair is known for the system under consideration, our approach cannot enter the general relaxation theory developped by Liu [47], Chen et al. [16]. We are thus led to adopt the framework proposed by Whitham [64] which relies on suitable Chapman-Enskog expansions. As expected, such expansions will provide us with some *sub-characteristic like conditions* expressing that some diffusion matrix must stay non-negative (see also [16] and [47]).

The scheme we propose below differs from the classical relaxation scheme developed by Jin and Xin [69] (recently studied by Natalini [52], Aregba and Natalini [3] in the scalar cases) since not all

the nonlinearities are relaxed. This partial relaxation procedure is more physically relevant and have been already proposed by various authors, in particular Jin & Slemrod [37, 38], Coquel & Perthame [20] and Coquel et al. [19]. In the present work, only the pressure law and the hydrodynamic one are subject to a relaxation procedure. Relaxing solely these quantities will considerably facilitate the Chapmann-Enskog like analysis. It will in particular unable us to exhibit precise and simple stability conditions for our relaxation model. Another feature of the scheme we propose is that positiveness for the partial densities is ensured. This is achieved by extending Larrouturou's ideas [40].

The present paper is organized as follow. In the next section, the relaxation model is proposed and its main properties are analyzed. This relaxation model involves two constant speeds of propagation (understood as free parameters in our procedure) which precise definition will be carried out on the ground of several stability requirements. In the second section, a fairly simple approximate Riemann solver is derived within the frame of relaxation methods. A special attention is paid on preserving the positiveness of the densities. With this respect, a suitable and simple definition of the two constant speeds of propagation of the model is shown to exist. Finally in the last section, some numerical results are discussed.

3.1 A relaxation model

In order to conveniently relax the main nonlinearities involved in the system under consideration, namely both the pressure law and the hydrodynamic one ; we first perform a conservative change of variables which thus preserves the weak solutions of (3.1)–(3.2). In that aim, let us consider the total density of the mixing $\rho = \rho_{\rm L} R_{\rm L} + \rho_{\rm G} R_{\rm G}$, the total momentum $\rho v = \rho_{\rm L} R_{\rm L} v_{\rm L} + \rho_{\rm G} R_{\rm G} v_{\rm G}$ and ρY where Y denotes the mass fraction of one of the two phases. To fix ideas and without restriction, we choose $Y = \frac{\rho_{\rm G} R_{\rm G}}{\rho}$ (so that $1 - Y = \frac{\rho_{\rm L} R_{\rm L}}{\rho}$). The natural phase space associated with such variables then reads :

$$\Omega = \left\{ \mathbf{u} = (\rho,\ \rho v,\ \rho Y) \in \mathbb{R}^3;\ \rho > 0,\ v \in \mathbb{R},\ Y \in [0,1] \right\}.$$

With some little abuse, both the pressure law p and the hydrodynamic closure Φ will keep their previous notations when expressed in terms of the new variable $\mathbf{u}$. In order to simplify the notations, let us introduce the following two functions of the unknown $\mathbf{u}$:

$$\sigma(\mathbf{u}) = \rho Y(1-Y)\Phi(\mathbf{u}), \quad P(\mathbf{u}) = p(\mathbf{u}) + \rho Y(1-Y)\Phi(\mathbf{u})^2. \tag{3.4}$$

Equipped with this admissible change of variables, we state the following easy result :

Lemma 3.1.1 *Weak solutions of the system (3.1)–(3.2) equivalently obey the following system in conservation form :*

$$\left\{ \begin{array}{lcl} \partial_t(\rho) + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + P(\mathbf{u})) = 0, \\ \partial_t(\rho Y) + \partial_x(\rho Y v - \sigma(\mathbf{u})) = 0. \end{array} \right. \tag{3.5}$$

To shorten the forthcoming notations, the system (3.5) will be given the condensed form :

$$\partial_t \mathbf{u} + \partial_x \mathcal{F}(\mathbf{u}) = 0, \quad t > 0,\ x \in \mathbb{R}; \tag{3.6}$$

where the flux function $\mathcal{F} : \Omega \to \mathbb{R}^3$ finds a clear definition. The system (3.6) will be referred as the *equilibrium* system in Eulerian coordinates.

PROOF The two functions σ and P clearly encode the most severe nonlinearities in the system (3.5). The first equation in (3.5) readily follows when adding the two mass conservation laws in (3.1). Straightforward calculations yield the following two identities $v_{\rm L} = v + Y\Phi(\mathbf{u})$ and $v_{\rm G} =$

$v-(1-Y)\Phi(\mathbf{u})$. The last equation in (3.5) henceforth follows. We conclude the equivalence statement arguing again from the above two identities to get :

$$\begin{aligned}\rho_{\mathrm{L}} R_{\mathrm{L}} v_{\mathrm{L}}^2 + \rho_{\mathrm{G}} R_{\mathrm{G}} v_{\mathrm{G}}^2 &= \rho(1-Y)\left(v + Y\Phi(\mathbf{u})\right)^2 + \rho Y\left(v-(1-Y)\Phi(\mathbf{u})\right)^2 \\ &= \rho v^2 + \rho Y(1-Y)\Phi(\mathbf{u})^2.\end{aligned} \tag{3.7}$$

which is the desired result. □

3.1.1 Design of the relaxation system

We propose to rewrite the system (3.5) in terms of the Lagrangian coordinates based on the density ρ and the velocity v. Let us set $\tau = 1/\rho$, and define the Lagrangian mass coordinate y setting $dy = \rho dx - \rho v dt$. Then the equilibrium system in Lagrangian coordinates (see for instance Serre [59] or Godlewski and Raviart [33]) writes

$$\left\{\begin{array}{lll}\partial_t \tau & - \partial_y v & = 0, \\ \partial_t v & + \partial_y P(\mathbf{u}) & = 0, \\ \partial_t Y & - \partial_y \sigma(\mathbf{u}) & = 0,\end{array}\right. \tag{3.8}$$

We propose to approximate the solutions of (3.5) by those of a relaxation system, obtained by solely relaxing the nonlinearities in $\sigma(\mathbf{u})$ and $P(\mathbf{u})$. The very motivation stems for the derivation of easily tractable sub-characteristic conditions as put forward below. We thus introduce two new state variables Σ and Π which are intended to coincide respectively with $\sigma(\mathbf{u})$ and $P(\mathbf{u})$ in the limit of some relaxation parameter, say λ. We propose as a relaxation model the following *nonlinear* first order system with singular perturbations :

$$\left\{\begin{array}{lll}\partial_t \tau^\lambda & - \partial_y v^\lambda & = 0, \\ \partial_t v^\lambda & + \partial_y \Pi^\lambda & = 0, \\ \partial_t \Pi^\lambda & + a^2\, \partial_y v^\lambda & = \lambda(P(\mathbf{u}) - \Pi^\lambda), \\ \partial_t Y^\lambda & - \partial_y \Sigma^\lambda & = 0, \\ \partial_t \Sigma^\lambda & - b^2\, \partial_y Y^\lambda & = \lambda(\sigma(\mathbf{u}) - \Sigma^\lambda).\end{array}\right. \tag{3.9}$$

Then we return in the original frame and consider the relaxation system in Eulerian coordinates

$$\left\{\begin{array}{lll}\partial_t (\rho)^\lambda & + \partial_x (\rho v)^\lambda & = 0, \\ \partial_t (\rho v)^\lambda & + \partial_x (\rho v^2 + \Pi)^\lambda & = 0, \\ \partial_t (\rho \Pi)^\lambda & + \partial_x (\rho \Pi v + a^2 v)^\lambda & = \lambda \rho^\lambda (P(\mathbf{u}^\lambda) - \Pi^\lambda), \\ \partial_t (\rho Y)^\lambda & + \partial_x (\rho Y v - \Sigma)^\lambda & = 0, \\ \partial_t (\rho \Sigma)^\lambda & + \partial_x (\rho \Sigma v - b^2 Y)^\lambda & = \lambda \rho^\lambda (\sigma(\mathbf{u}^\lambda) - \Sigma^\lambda).\end{array}\right. \tag{3.10}$$

For simplicity in the notations, the superscript λ will be omitted most of the time. The next statement will be useful in the forthcoming developments :

Lemma 3.1.2 *Smooth solutions of (3.10) satisfy the following system in non conservation form :*

$$\left\{\begin{array}{l}\partial_t \tau + v\partial_x \tau - \tau \partial_x v = 0, \\ \partial_t v + v\partial_x v + \tau \partial_x \Pi = 0, \\ \partial_t \Pi + v\partial_x \Pi + a^2 \tau \partial_x v = \lambda(P(\mathbf{u}) - \Pi), \\ \partial_t Y + v\partial_x Y - \tau \partial_x \Sigma = 0, \\ \partial_t \Sigma + v\partial_x \Sigma - b^2 \tau \partial_x Y = \lambda(\sigma(\mathbf{u}) - \Sigma).\end{array}\right. \tag{3.11}$$

The relaxation system (3.10) will be given hereafter the convenient abstract form :

$$\partial_t \mathbf{v} + \partial_x \mathcal{G}(\mathbf{v}) = \lambda \mathcal{R}(\mathbf{v}), \quad t > 0,\ x \in \mathbb{R}; \tag{3.12}$$

where both the flux function $\mathcal{G}$ and the relaxation terms $\mathcal{R}$ receive clear definitions. The associated admissible state space $\mathcal{V}$ reads

$$\begin{aligned} \mathcal{V} = \big\{ \mathbf{v} = {}^t(\rho,\ \rho v,\ \rho\Pi,\ \rho Y,\ \rho\Sigma) \in \mathbb{R}^5; \\ \rho > 0,\ v \in \mathbb{R},\ \Pi \in \mathbb{R},\ Y \in [0,1],\ \Sigma \in \mathbb{R} \big\}. \end{aligned} \tag{3.13}$$

In (3.10), a and b denote two real free parameters in the relaxation procedure we propose. All along the present work, we shall pay a central attention to the specification of these two constants, on the ground of several stability requirements. With this respect, successive and sharper definitions for such a pair will be given in the forthcoming sections. The final definition of (a, b) is postponed to Section 3.2.4. It is however useful to shed some light, from now on, on the significance of the pair (a, b).

The proof of this easy result is left to the reader. Let us observe that the Lagrangian form (3.9) of our relaxation model is nothing else a first order *quasi-linear* system with singular perturbations. With this respect, the relaxation procedure we propose now finds clear relationships with the Xin and Jin approach [69] but at the expense of a Lagrangian transformation. To go further, let us set the relaxation parameter λ to zero. Then, the system (3.9) splits itself into two independent linear hyperbolic systems, one in the unknowns (τ, v, Π), the other in (Y, Σ). The first group of unknowns can be associated respectively with the specific volume, the velocity and the pressure of an hypothetic gas where the parameter a would play the role of a (constant) Lagrangian sound speed. Likewise, the second pair would characterize the specific volume Y and the velocity Σ of a hypothetic isentropic gas with the parameter b as sound speed. These two (linear) hypothetic gas dynamics systems have connections with the recent works by Bouchut [13], Després [22] and Suliciu [60] ; all these works being primarily devoted to the Lagrangian setting. Our approach, already introduced in other contexts (see [20] and [19]), could be thus understood as an extension to the Eulerian framework.

Let us now state the basic properties of the Eulerian form (3.10).

Lemma 3.1.3 *Let (a, b) be a pair of strictly positive real numbers For any $\mathbf{v} \in \mathcal{V}$, the first order system extracted from (3.10) admits five real eigenvalues $(\tau = 1/\rho)\ v,\ v \pm a\tau,\ v \pm b\tau$, and five linearly independent corresponding eigenvectors. Consequently, the first order extracted system from (3.10) is hyperbolic on $\mathcal{V}$. Moreover, each eigenvalue is associated with a linearly degenerate field.*

Linear degeneracy of each of the fields is a by-product of the derivation principle of the nonlinear relaxation model (3.10). Such a property is expected from the Lagrangian form (3.9) (see indeed Wagner [63]). It is actually desired since it makes straightforwardly solvable the Riemann problem associated with (3.10) when $\lambda = 0$ (see indeed Proposition 3.2.2, Section 3.2.1).

PROOF The hyperbolicity properties of a first order nonlinear system are known to be independent from a given change of variables (see Godlewski and Raviart [33] for instance). It is then convenient to use the non conservation form (3.11) to immediately derive the required eigenvalues. Easy calculations show that the following vectors $r_i(\mathbf{v})$, $i = 1, \ldots, 5$, given by :

$$(1, 0, 0, 0, 0), \quad (\rho, \pm a\tau, a^2\tau, 0, 0) \quad \text{and} \quad (0, 0, 0, 1, \mp b), \tag{3.14}$$

are respectively right eigenvectors for the eigenvalues $\lambda_i(\mathbf{v}) : v,\ v \pm a\tau$, and $v \pm b\tau$. These vectors are clearly independent provided that the parameters a and b are not zero. It is easily seen that $\nabla\lambda_i(\mathbf{v}) \cdot r_i(\mathbf{v}) = 0$ for $i = 1, \ldots, 5$ for all $\mathbf{v} \in \mathcal{V}$. Such a property expresses, after Lax [43], the linear degeneracy of all the fields under consideration. □

3.1.2 Chapman-Enskog expansion

The properties stated in Lemma 3.1.3 holds true regardless of the choice of a non zero pair (a, b) (say strictly positive for definiteness). However, it is known after the works by Liu [47], Chen, Levermore and Liu [16] that some compatibility conditions must be satisfied by the original system (3.1) (we shall refer as to the equilibrium system in the sequel) and its relaxation approximation (3.10). These conditions are actually needed to prevent the relaxation approximation procedure from instabilities as λ goes to infinity. They are usually referred as to sub-characteristic like conditions after Whitham [64]. It is therefore expected that the parameters a and b must be fixed in order to fulfill such stability requirements. There exists a variety of ways to exhibit the needed conditions. A powerful one requires the existence of compatible Lax entropy pairs for both the equilibrium system and its relaxation approximation (see Chen, Levermore and Liu [16] for the details). However and as already claimed, the equilibrium system under consideration (3.1) fails to admit additional non trivial conservation laws for general hydrodynamic closure (3.2). This lack for equilibrium entropy pair thus forbids us to enter the general framework proposed in [16].

These authors have proposed (see also Whitham [64]) a weaker approach based on the derivation of the first order asymptotic equilibrium system. In our relaxation framework, its derivation is based on a Chapman-Enskog expansion of small departures $\mathbf{v}^\lambda$ from the local equilibrium $\mathbf{u}$:

$$\Pi^\lambda = P(\mathbf{u}^\lambda) + \lambda^{-1}\Pi_1^\lambda + \mathcal{O}\left(\lambda^{-2}\right), \tag{3.15a}$$

$$\Sigma^\lambda = \sigma(\mathbf{u}^\lambda) + \lambda^{-1}\Sigma_1^\lambda + \mathcal{O}\left(\lambda^{-2}\right). \tag{3.15b}$$

In (3.15), the dependency of the first order correctors Π_1^λ and Σ_1^λ in λ comes from the fact that they actually depend on both the equilibrium unknown $\mathbf{u}^\lambda$ and its space derivative. After substituting (3.15) into (3.10) and neglecting higher order terms, we classically end up with the first order asymptotic equilibrium system with the following generic form (see Proposition 3.1.5 for a brief derivation) :

$$\partial_t \mathbf{u}^\lambda + \partial_x \mathcal{F}(\mathbf{u}^\lambda) = \lambda^{-1}\partial_x(\mathcal{D}_{(a,b)}(\mathbf{u}^\lambda)\partial_x \mathbf{u}^\lambda), \tag{3.16}$$

where the flux function $\mathcal{F}$ coincides with the one in (3.6) and where the tensor $\mathcal{D}_{(a,b)}$ will be given an explicit form hereafter. According to the notations, this tensor does actually depend on the free parameters a and b.

Now, the stability conditions to be put on the pairs (a, b) clearly come from the requirement that the first-order correction operator in (3.16) must be dissipative relatively to the zero-order approximation (3.6). Such conditions may be obtained by establishing the L^2-stability of the constant coefficient problem obtained by linearizing (3.16) in the neighborhood of any equilibrium state $\mathbf{u}$. In other words, for all admissible states $\mathbf{u}$, all eigenvalues of the matrix $+i\xi\nabla_{\mathbf{u}}\mathcal{F}(\mathbf{u}) - \lambda^{-1}\xi^2\mathcal{D}_{(a,b)}(\mathbf{u})$ for all $\xi \in \mathbb{R}$, should have negative real parts (see Serre [59] for instance). The algebraic complexity of the Jacobian matrix $\nabla_{\mathbf{u}}\mathcal{F}$ (for general hydrodynamic closures (3.2)) virtually makes intractable such a criterion for our numerical purposes. As a consequence, we adopt the weaker condition on (a, b) according to which the eigenvalues of the dissipative tensor $\mathcal{D}_{(a,b)}(\mathbf{u})$ must be non-negative real numbers for all the states $\mathbf{u}$ under consideration. The main result of this section actually shows that it is always possible to fulfill this stability requirement.

Proposition 3.1.4 *There exist four smooth mappings A, B, C, $D : \Omega \to \mathbb{R}$ (explicitly known and detailed below) such that for any given $\mathbf{u} \in \Omega$, the tensor $\mathcal{D}_{(a,b)}$ meets the form :*

$$\mathcal{D}_{(a,b)}(\mathbf{u}) = \frac{1}{\rho^2}\begin{pmatrix} 0 & 0 & 0 \\ \times & a^2 - A(\mathbf{u}) & C(\mathbf{u}) \\ \times & D(\mathbf{u}) & b^2 - B(\mathbf{u}) \end{pmatrix}. \tag{3.17}$$

This tensor therefore always admits one zero eigenvalue. For any given $\mathbf{u} \in \Omega$*, the two other non trivial eigenvalues of* $\mathcal{D}_{(a,b)}(\mathbf{u})$ *are positive iff the following set of inequalities is satisfied by the pair* (a, b) *:*

$$\begin{array}{rcl} \left[\left(a^2 - A(\mathbf{u})\right) - \left(b^2 - B(\mathbf{u})\right)\right]^2 & \geq & -4C(\mathbf{u})D(\mathbf{u}), \\ \left(a^2 - A(\mathbf{u})\right)\left(b^2 - B(\mathbf{u})\right) & \geq & C(\mathbf{u})D(\mathbf{u}), \\ \left(a^2 - A(\mathbf{u})\right) + \left(b^2 - B(\mathbf{u})\right) & \geq & 0. \end{array} \tag{3.18}$$

Let us observe that whatever the signs of the bounded real numbers $A(\mathbf{u})$, $B(\mathbf{u})$, $C(\mathbf{u})$ and $D(\mathbf{u})$ may be for a given $\mathbf{u} \in \Omega$, the parameters a and b can be always chosen large enough so that (3.18) is valid. We shall give below a sharper and more convenient characterization of admissible pairs (a, b).

Assuming first that the tensor $\mathcal{D}_{(a,b)}$ meets the above mentioned form, the set of inequalities (3.18) obviously comes from the fact that the two non trivial eigenvalues of $\mathcal{D}_{(a,b)}(\mathbf{u})$ are roots of the quadratic equation

$$\begin{array}{l} \lambda^2 - \left[a^2 - A(\mathbf{u}) + b^2 - B(\mathbf{u})\right]\lambda \\ + \left[a^2 - A(\mathbf{u})\right]\left[b^2 - B(\mathbf{u})\right] - C(\mathbf{u})D(\mathbf{u}) = 0. \end{array}$$

Requiring these roots to be positive real numbers is equivalent first to enforce the discriminant of this equation to remain non negative (hence the first inequality in (3.18)) and in addition to enforce both the sum and the product of the roots to be non negative (hence respectively the two last inequalities).

The derivation of the precise shape of $\mathcal{D}_{(a,b)}(\mathbf{u})$ relies on easy but cumbersome algebraic calculations. It turns out that expressing the main nonlinearities of the problem in terms of the variables (τ, v, Y) instead of the conservative one leads to simpler calculations. For the sake of simplicity and with some abuse in the notations, nonlinear functions (like the pressure or the hydrodynamic law) will be given the same notation when expressed in both types of variables. The final results of the calculations are gathered in the following proposition :

Proposition 3.1.5 *The first order asymptotic equilibrium system (3.16) reads*

$$\left\{ \begin{array}{lllll} \partial_t\left(\rho\right)^\lambda & + & \partial_x\left(\rho v\right)^\lambda & = & 0, \\ \partial_t\left(\rho v\right)^\lambda & + & \partial_x\left(\rho v^2 + P(\mathbf{u})\right))^\lambda & = & \lambda^{-1}\partial_x\left(\mathbf{D}^1(\mathbf{u}^\lambda)\cdot\partial_x\mathbf{u}^\lambda\right), \\ \partial_t\left(\rho Y\right)^\lambda & + & \partial_x\left(\rho Y v - \sigma(\mathbf{u})\right)^\lambda & = & \lambda^{-1}\partial_x\left(\mathbf{D}^2(\mathbf{u}^\lambda)\cdot\partial_x\mathbf{u}^\lambda\right). \end{array} \right. \tag{3.19}$$

To define the smooth mappings $\mathbf{D}^1$, $\mathbf{D}^2 : \Omega \to \mathbb{R}^3$*, let us set*

$$C(\mathbf{u}) = -P_v(\mathbf{u})\, P_Y(\mathbf{u}) \;+\; P_Y(\mathbf{u})\, \sigma_Y(\mathbf{u}), \tag{3.20}$$

$$D(\mathbf{u}) = -\,\sigma_\tau(\mathbf{u}) \;+\; \sigma_v(\mathbf{u})\, P_v(\mathbf{u}) \;-\; \sigma_Y(\mathbf{u})\, \sigma_v(\mathbf{u}), \tag{3.21}$$

$$A(\mathbf{u}) = -P_\tau(\mathbf{u}) \;+\; P_v^2(\mathbf{u}) \;-\; P_Y(\mathbf{u})\, \sigma_v(\mathbf{u}), \tag{3.22}$$

$$B(\mathbf{u}) = \;\sigma_Y^2(\mathbf{u}) \;-\; \sigma_v(\mathbf{u})\, P_Y(\mathbf{u}) \tag{3.23}$$

where we have noted the partial derivatives with subscripts (for example $P_\tau(\mathbf{u}) = \frac{\partial P}{\partial \tau}(\mathbf{u})$*). Then,*

$$\mathbf{D}^1(\mathbf{u}) = \frac{1}{\rho^2}\begin{pmatrix} \times \\ a^2 - A(\mathbf{u}) \\ C(\mathbf{u}) \end{pmatrix} \qquad \textit{and} \qquad \mathbf{D}^2(\mathbf{u}) = \frac{1}{\rho^2}\begin{pmatrix} \times \\ D(\mathbf{u}) \\ b^2 - B(\mathbf{u}) \end{pmatrix}.$$

Remark 3.1.6 *The* σ_Y^2 *term in* B *is the one which appears in the relaxation of the scalar equation* $\partial_t Y - \partial_y\tilde{\sigma}(Y) = 0$*. The* P_τ *term in* A *is the one which appears in the relaxation of the Isothermal Euler system. This is natural since the Lagrangian structure of the system (3.5) is (3.8) which is close to this two models. The others terms which appear in (3.20–3.23) come therefore from the coupling between the equations.*

PROOF Considering smooth solutions of the relaxation system (3.10), the relaxed pressure Π and the velocity Σ are easily seen to satisfy

$$\begin{array}{l} \partial_t \Pi + v\partial_x \Pi + a^2 \tau \partial_x v = \lambda(P(\mathbf{u}) - \Pi), \\ \partial_t \Sigma + v\partial_x \Sigma - b^2 \tau \partial_x Y = \lambda(\sigma(\mathbf{u}) - \Sigma). \end{array}$$

Plugging the expansions (3.15) into the two above equations and dropping the higher order terms, we end up with

$$\begin{array}{l} \nabla_{\mathbf{u}} P(\mathbf{u})\partial_t \mathbf{u} + v\partial_x P(\mathbf{u}) + a^2\tau\partial_x v = -\Pi_1 + \mathcal{O}\left(\lambda^{-1}\right), \\ \nabla_{\mathbf{u}} \sigma(\mathbf{u})\partial_t \mathbf{u} + v\partial_x \sigma(\mathbf{u}) - b^2\tau\partial_x Y = -\Sigma_1 + \mathcal{O}\left(\lambda^{-1}\right). \end{array} \tag{3.24}$$

In the above equations, we wish to turn time derivatives into space derivatives. In that aim, it suffices to express $\partial_t \mathbf{u}$ in terms of the time derivatives of ρ, v and Y and then to use the following identities :

$$\left\{ \begin{array}{l} \partial_t \rho = -v\partial_x \rho - \rho\partial_x v, \\ \partial_t v = -v\partial_x v - \tau\partial_x P(\mathbf{u}), \\ \partial_t Y = -v\partial_x Y + \tau\partial_x \sigma(\mathbf{u}), \end{array} \right. \tag{3.25}$$

which are similar to those stated in Lemma 3.1.2. Using these equalities in (3.24) yields after lengthy calculations (which we shall not not report here)

$$\Pi_1 = -\mathbf{D}^1(\mathbf{u}) \cdot \partial_x \mathbf{u}, \qquad \Sigma_1 = +\mathbf{D}^2(\mathbf{u}) \cdot \partial_x \mathbf{u}. \tag{3.26}$$

To achieve the proof, let us consider the following set of PDE's extracted from the relaxation model (3.10) :

$$\left\{ \begin{array}{lll} \partial_t (\rho)^\lambda + \partial_x (\rho v^\lambda) & = & 0, \\ \partial_t (\rho v)^\lambda + \partial_x (\rho v^2 + \Pi)^\lambda & = & 0, \\ \partial_t (\rho Y)^\lambda + \partial_x (\rho Y v - \Sigma)^\lambda & = & 0. \end{array} \right. \tag{3.27}$$

It is then sufficient to plug the expansions (3.15) into these equations to obtain

$$\left\{ \begin{array}{lll} \partial_t (\rho)^\lambda + \partial_x (\rho v)^\lambda & = & 0, \\ \partial_t (\rho v)^\lambda + \partial_x (\rho v^2 + P(\mathbf{u}))^\lambda & = & -\lambda^{-1}\partial_x \Pi_1 + \mathcal{O}\left(\lambda^{-2}\right), \\ \partial_t (\rho Y)^\lambda + \partial_x (\rho Y v - \sigma(\mathbf{u}))^\lambda & = & \lambda^{-1}\partial_x \Sigma_1 + \mathcal{O}\left(\lambda^{-2}\right), \end{array} \right. \tag{3.28}$$

which is nothing but the expected system (3.19), thanks to (3.26). □

To conclude this section, let us notice that a sharp characterization for a given $\mathbf{u} \in \Omega$ of all the pairs (a, b) satisfying the stability condition (3.18) obviously depends on the sign of $A(\mathbf{u})$, $B(\mathbf{u})$, $C(\mathbf{u})$ and $D(\mathbf{u})$. For numerical purposes, it seems convenient to propose a characterization of admissible pairs which stays free from signs consideration. In that aim, let us prove the following lemma.

Lemma 3.1.7 *Let us define $\gamma(\mathbf{u}) = |C(\mathbf{u})D(\mathbf{u})|$. Let us define the couple $(\widehat{a}_-, \widehat{b}_-)$ by*

$$\widehat{a}_-(\mathbf{u}) = (\sqrt{2}+1)\sqrt{\gamma(\mathbf{u})}, \quad \widehat{b}_-(\mathbf{u}) = (\sqrt{2}-1)\sqrt{\gamma(\mathbf{u})}, \tag{3.29}$$

and the couple (a_-, b_-) by

$$a_-(\mathbf{u}) = \sqrt{\widehat{a}_-(\mathbf{u}) + A(\mathbf{u})}, \quad b_- = \sqrt{\widehat{b}_-(\mathbf{u}) + B(\mathbf{u})}. \tag{3.30}$$

Then the conditions (3.18) are satisfied.

PREUVE The idea consists in considering the change of variables :

$$\widehat{a} = a^2 - A, \qquad \widehat{b} = b^2 - B, \tag{3.31}$$

where, now, the unknowns are $\widehat{a}, \widehat{b}$ and not a, b anymore. The conditions (3.18) now become :

$$\left(\widehat{a} - \widehat{b}\right)^2 + 4\,CD \;\geq\; 0, \tag{3.32}$$

$$\widehat{a}\widehat{b} - CD \;\geq\; 0, \tag{3.33}$$

$$\widehat{a} + \widehat{b} \;\geq\; 0. \tag{3.34}$$

It is easy to see that a sufficient condition for (3.32–3.34) is :

$$\left(\widehat{a} - \widehat{b}\right)^2 - 4\,\gamma \;\geq\; 0, \tag{3.35}$$

$$\widehat{a}\widehat{b} - \gamma \;\geq\; 0, \tag{3.36}$$

$$\widehat{a} + \widehat{b} \;\geq\; 0, \tag{3.37}$$

with $\gamma = |CD|$. On the figure (3.1), we can see the solutions of this system of inequalities in the plan $(\widehat{a}, \widehat{b})$.

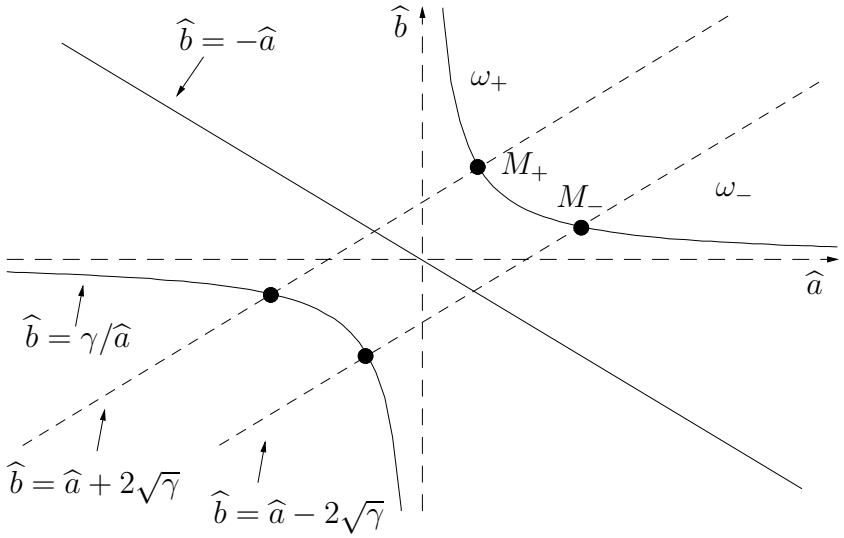

FIG. 3.1 – Domains $\omega_+(\mathbf{u})$ and $\omega_-(\mathbf{u})$

We see that the equalities associated with the inequalities (3.35) and (3.36) have always 4 intersections but the inequality (3.37) pick up only two points among them : we noted them M_+ and M_-. We define by ω_+ (resp. ω_-) the domain associated with M_+ (resp. M_-). In these two domains, which are defining all the solutions of the system (3.35–3.37), we choose the couples $(\widehat{a},\ \widehat{b})$ so that they are as small as possible in order to minimize the numerical dissipation. These points are precisely the points M_+ and M_-.

It is simple to compute the coordinates of the point M_- because it is so that $\widehat{a}$ is the positive solution of the equation $\frac{\gamma}{\widehat{a}} = \widehat{a} - 2\sqrt{\gamma}$ which can be written $\widehat{a}^2 - 2\sqrt{\gamma}\ \widehat{a} - \gamma = 0$. Therefore, the coordinates of M_- are :

$$\widehat{a}_- = \left(\sqrt{2}+1\right)\sqrt{\gamma}, \qquad \widehat{b}_- = \left(\sqrt{2}-1\right)\sqrt{\gamma}. \tag{3.38}$$

In the same way, the coordinates of M_+ are :

$$\widehat{a}_+ = \left(\sqrt{2}-1\right)\sqrt{\gamma}, \qquad \widehat{b}_+ = \left(\sqrt{2}+1\right)\sqrt{\gamma}. \tag{3.39}$$

We clearly see that $\widehat{a}_- > \widehat{b}_-$ whereas $\widehat{a}_+ < \widehat{b}_+$. But we are primarily interested by the slow waves that is why we prefer to minimize $\widehat{b}$: our choice is the couple $(\widehat{a}_-,\ \widehat{b}_-)$.□

In the next section devoted to the numerical approximation of the solutions of the equilibrium system (3.5), the above sets $\omega_+(\mathbf{u})$ and $\omega_-(\mathbf{u})$ will play an important role.

3.2 An approximate Riemann solver

In this section we propose an approximate Riemann solver for the equilibrium system (3.1). As already pointed out, the companion relaxation system (3.10) only admits linearly degenerate fields. As we shall see, the Riemann problem associated with this system can be trivially solved. With such a benefit, we propose below a relaxation scheme based on the Godunov method for (3.10), when considering parameters a and b satisfying the stability conditions we have put forward in Lemma 3.1.7. In addition to simplicity, using the exact Riemann solver but for (3.10) will allow us to easily enforce both the positivity of the density in addition to a maximum principle on the mass fractions ($Y \in [0,1]$) for a suitable choice of the free parameters a and b.

The numerical procedure we use, is standard within the frame of relaxation methods (see for instance Xin and Jin [69] ..) but we briefly recall it for the sake of completeness. We begin this section with a rough description of the numerical procedure we develop below for approximating the weak solutions of (3.10) within the frame of the hydrodynamic relaxation theory. Given some approximation of the equilibrium solution at the time t^n, say $\mathbf{u}^n(x) = {}^t(\rho^n, (\rho v)^n, (\rho Y)^n)(x)$, this approximation is evolved at the next time level $t^{n+1} = t^n + \Delta t$ into two steps.

1. *Relaxation*
   At time $t = t^n$, we solve the ODE system $d_t\mathbf{u} = \lambda\mathcal{R}(\mathbf{u})$ with $\lambda \to \infty$ and set $\Pi^n(x) = P(\mathbf{u}^n(x))$, $\Sigma^n(x) = \sigma(\mathbf{u}^n(x))$.
2. *Evolution in time*
   We take $\lambda = 0$ and solve the system of partial differential equations $\partial_t\mathbf{v} + \partial_x\mathcal{G}(\mathbf{v}) = 0$ in order to go to time $t^{n+1} = t^n + \Delta t$.

Remark 3.2.1 *In [69], Jin & Xin make a distinction between the "relaxing scheme" in which one uses a small fixed value of $\varepsilon = \frac{1}{\lambda}$ (for example $\varepsilon = 10^{-8}$) and the relaxed scheme which is the $\varepsilon = 0$ limit of the relaxing schemes. Our scheme is of the second class.*

3.2.1 Relaxation scheme

We now give a complete description of the numerical method we propose. Let Δt and Δx be respectively the time and space steps, chosen to be uniform for simplicity. Let us define the time levels $t^n = n\Delta t$ for $n \in \mathbb{N}$ and the cell interface location $x_{i+1/2} = (i+1/2)\Delta x$ with $i \in \mathbb{Z}$. We consider piecewise constant approximate equilibrium solutions $\mathbf{u}^h(x,t) : \mathbb{R}\times\mathbb{R}_+ \mapsto \Omega$ under the classical form :

$$\begin{aligned}\mathbf{u}^h(x,t) = \mathbf{u}_i^n = {}^T(\rho_i^n,\ (\rho v)_i^n,\ (\rho Y)_i^n)\,,\\ (x,t) \in]x_{i-1/2}, x_{i+1/2}[\times[t^n, t^{n+1}[.\end{aligned} \tag{3.40}$$

At $t = 0$, we set

$$\mathbf{u}_i^0 = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} {}^T\big(\rho^0(x),\ (\rho v)^0(x),\ (\rho Y)^0(x)\big)\ dx.$$

In order to advance the approximate equilibrium solution in time, we define another function $\mathbf{v}^h(x,t) : \mathbb{R}\times\mathbb{R}_+ \mapsto \mathcal{V}$ which is also piecewise constant at each time level t^n. In each slab $\mathbb{R}\times\,[t^n, t^{n+1}[$, the function $\mathbf{v}^h(x, t^n + t)$, with $0 < t < \Delta t$, is the weak solution of the Cauchy problem for (3.10) obtained with $\lambda = 0$ while prescribing the following initial data given for $x \in]x_{i-1/2}, x_{i+1/2}[$:

$$\mathbf{v}^h(x,t^n) = \mathbf{v}_i^n = {}^t((\rho)_i^n,\ (\rho v)_i^n,\ (\rho\Pi)_i^n,\ (\rho Y)_i^n,\ (\rho\Sigma)_i^n) \tag{3.41}$$

with $(\rho\Pi)_i^n := \rho_i^n P(\mathbf{u}_i^n)$ and $(\rho\Sigma)_i^n := \rho_i^n \sigma(\mathbf{u}_i^n)$. When Δt is small enough, e.g. under the CFL condition

$$\frac{\Delta t}{\Delta x}\max|\mu_i(\mathbf{v}^h)| \leq \frac{1}{2}, \tag{3.42}$$

where $(\mu_i)_{1\leq i\leq 5}$ denotes the eigenvalues defined in Lemma 3.1.3, the function $\mathbf{v}^h$ is classically obtained by solving a sequence of local Riemann problems without interaction, located at each cell interface $x_{i+1/2}$.

The convenient CFL restriction (3.42) will allow us to use the celebrated Harten, Lax and Van Leer [35] formalism which will turn out to be particularly well-suited to our forthcoming purpose : namely a local definition of the pair (a,b) at each cell interface $x_{i+1/2}$ (see below and Coquel and Perthame [20]). Following these authors, let be given two equilibrium states $\mathbf{u_L}$ and $\mathbf{u_R}$ in Ω and define $\mathbf{v_L} = \mathbf{v}(\mathbf{u_L})$ and $\mathbf{v_R} = \mathbf{v}(\mathbf{u_R})$ according to (3.41). We may think of choosing $\mathbf{v_L} = \mathbf{v}_i^n$ and $\mathbf{v_R} = \mathbf{v}_{i+1}^n$ when considering the interface $x_{i+1/2}$. Let us choose a pair of parameters (a,b) for completing the definition of the relaxation system (3.10). Precise conditions on these parameters will be introduced later on. Let us keep in mind that the pair under consideration is to be labeled with reference to the given interface $x_{i+1/2}$.

Now, equipped with such a pair, let $w_{a,b}(.,\mathbf{v_L},\mathbf{v_R})$ denote the solution of the Riemann problem for $(3.10)_{a,b}$ with initial data $\mathbf{v}_0(x) = \mathbf{v_L}$ if $x < 0$ and $\mathbf{v_R}$ otherwise. We easily get after Harten, Lax and Van Leer [35] :

$$\begin{aligned}\bar{\mathbf{v}}_{\mathbf{L}}(\mathbf{v_L},\mathbf{v_R}) &:= \frac{2\Delta t}{\Delta x}\int_{-\frac{\Delta x}{2\Delta t}}^{0} w_{a,b}(\xi,\mathbf{v_L},\mathbf{v_R})d\xi \\ &= \mathbf{v_L} - \frac{2\Delta t}{\Delta x}\left(\mathcal{G}_{a,b}(w_{a,b}(0^+;\mathbf{v_L},\mathbf{v_R})) - \mathcal{G}_{a,b}(\mathbf{v_L})\right),\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\bar{\mathbf{v}}_{\mathbf{R}}(\mathbf{v_L},\mathbf{v_R}) &:= \frac{2\Delta t}{\Delta x}\int_{0}^{\frac{\Delta x}{2\Delta t}} w_{a,b}(\xi,\mathbf{v_L},\mathbf{v_R})d\xi \\ &= \mathbf{v_R} - \frac{2\Delta t}{\Delta x}\left(\mathcal{G}_{a,b}(\mathbf{v_R}) - \mathcal{G}_{a,b}(w_{a,b}(0^+;\mathbf{v_L},\mathbf{v_R}))\right).\end{aligned} \tag{3.44}$$

In (3.43) and (3.44), the notation $\mathcal{G}_{a,b}$ obviously refers to exact flux function of the relaxation system (3.12) when labeled by the pair (a,b). Using clear notations, it is crucial to notice at this stage the following identities :

$$\mathcal{G}_{a,b}^{\rho}(\mathbf{v_L}) = (\rho v)_{\mathbf{L}}, \quad \mathcal{G}_{a,b}^{\rho v}(\mathbf{v_L}) = (\rho v^2)_{\mathbf{L}} + P(\mathbf{u_L}), \tag{3.45a}$$

$$\mathcal{G}_{a,b}^{\rho Y}(\mathbf{v_L}) = (\rho Y)_{\mathbf{L}} - \sigma(\mathbf{u_L}), \tag{3.45b}$$

while symmetrically

$$\mathcal{G}_{a,b}^{\rho}(\mathbf{v_R}) = (\rho v)_{\mathbf{R}}, \quad \mathcal{G}_{a,b}^{\rho v}(\mathbf{v_R}) = (\rho v^2)_{\mathbf{R}} + P(\mathbf{u_R}), \tag{3.46a}$$

$$\mathcal{G}_{a,b}^{\rho Y}(\mathbf{v_R}) = (\rho Y)_{\mathbf{R}} - \sigma(\mathbf{u_R}). \tag{3.46b}$$

The validity of (3.45)–(3.46) is directly inherited from the fact that $\mathbf{v_L}$ and $\mathbf{v_R}$ are respectively defined from the equilibrium states $\mathbf{u_L}$ and $\mathbf{u_R}$ (indeed see (3.41)). Put in other words, the identities (3.45)-(3.46) stay **completely free** from a particular choice of the pair (a, b) !

Equipped with these formulas, we define under the CFL condition (3.42)

$$\begin{aligned}\mathbf{v}_i^{n+1,-} &= \frac{1}{2}\left(\bar{\mathbf{v}}_{\mathbf{R}}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n) + \bar{\mathbf{v}}_{\mathbf{L}}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n)\right), \\ &= \left(\rho_i^{n+1,-}, (\rho v)_i^{n+1,-}, (\rho \Pi)_i^{n+1,-}, (\rho Y)_i^{n+1,-}, (\rho \Sigma)_i^{n+1,-}\right). \end{aligned} \tag{3.47}$$

The approximate equilibrium solution $\mathbf{u}^h(x, t^n)$ can be now advanced at the time level t^{n+1}, setting in each cell $]x_{i-1/2}, x_{i+1/2}[$:

$$\mathbf{u}^h(x, t^{n+1}) = \mathbf{u}_i^{n+1} = \left\{ \begin{array}{l} \rho_i^{n+1} = \rho_i^{n+1,-} \\ (\rho v)_i^{n+1} = (\rho v)_i^{n+1,-} \\ (\rho Y)_i^{n+1} = (\rho Y)_i^{n+1,-} \end{array} \right. \tag{3.48}$$

To summarize, as a consequence of the identities (3.45)-(3.46) and the formula definition (3.47), the updating formula for the equilibrium approximated solution reads :

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right),$$

where the numerical flux $\mathcal{F}_{i+1/2}^n$ is defined as

$$\begin{aligned}\mathcal{F}_{i+1/2}^n &= \mathcal{F}(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n) \\ &:= \left(\begin{array}{l} \mathcal{G}_{a,b}^{\rho}\left(w_{a_{i+1/2},b_{i+1/2}}\left(0^+; \mathbf{v}(\mathbf{u}_i^n), \mathbf{v}(\mathbf{u}_{i+1}^n)\right)\right) \\ \mathcal{G}_{a,b}^{\rho v}\left(w_{a_{i+1/2},b_{i+1/2}}\left(0^+; \mathbf{v}(\mathbf{u}_i^n), \mathbf{v}(\mathbf{u}_{i+1}^n)\right)\right) \\ \mathcal{G}_{a,b}^{\rho Y}\left(w_{a_{i+1/2},b_{i+1/2}}\left(0^+; \mathbf{v}(\mathbf{u}_i^n), \mathbf{v}(\mathbf{u}_{i+1}^n)\right)\right) \end{array} \right) \end{aligned}$$

The numerical approximation procedure we have just detailed, therefore recasts under the usual form of a finite volume method. Let us again underline that the definition of the pair $(a_{i+1/2}, b_{i+1/2})$ entering the numerical flux function (3.49) may vary from one interface to another. This again simply comes from the CFL condition (3.42).

To conclude the presentation of the relaxation method, we now have to exhibit the exact solution of the Riemann problem for the relaxation system (3.10) with $\lambda = 0$. Since all the fields of the system under consideration are linearly degenerate, the Riemann solution is systematically made of six constant states separated by five contact discontinuities. These states will be denoted in the sequel $\mathbf{v}_0 = \mathbf{v_L}$, $\mathbf{v}_j$, $j = 1, .., 4$, $\mathbf{v}_5 = \mathbf{v_R}$, where $(\mathbf{v_L}, \mathbf{v_R})$ denotes the pair of constant states in $\mathcal{V}$ defining the initial data

$$\mathbf{v}_0(x) = \left\{ \begin{array}{ll} \mathbf{v_L} & \text{if } x < 0, \\ \mathbf{v_R} & \text{if } x > 0. \end{array} \right. \tag{3.49}$$

The speed at which propagates the *j-th* contact discontinuity is thus given by $\mu_j(\mathbf{v}_j)$ for $j = 0, .., 4$ where the eigenvalues defined in Lemma 3.1.3 are tacitly assumed to be increasingly ordered. Again by the linear degeneracy of the fields, let us stress out that no entropy condition is needed to select the relevant Riemann solution (see for instance Godlewski and Raviart [33]). The Riemann solution thus always reads :

$$\mathbf{v}(x, t) = \left\{ \begin{array}{ll} \mathbf{v_L} & \text{if } \frac{x}{t} < \mu_1, \\ \mathbf{v}_j & \text{if } \mu_j < \frac{x}{t} < \mu_{j+1}, \quad 1 \leq j \leq 4 \\ \mathbf{v_R} & \text{if } \frac{x}{t} > \mu_5, \end{array} \right.$$

The following statement gives the intermediate states $\mathbf{v}_j$ for $j = 1, .., 4$. For simplicity, we shall only address the generic case $a \neq b$. In the following proposition, we note $\overline{\varphi} = (\varphi_{\mathbf{L}} + \varphi_{\mathbf{R}})/2$ the average and $\langle\varphi\rangle = (\varphi_{\mathbf{L}} - \varphi_{\mathbf{R}})/2$ the mid-jump of the the two states $\mathbf{v_L}$ and $\mathbf{v_R}$ whatever the quantity φ.

Proposition 3.2.2 *Let us set*

$$\begin{array}{rclrcl} \tau^\star_{\mathbf{L}} & = & \tau_{\mathbf{L}} - \frac{\langle v\rangle}{a} + \frac{\langle\Pi\rangle}{a^2}, & \tau^\star_{\mathbf{R}} & = & \tau_{\mathbf{R}} - \frac{\langle v\rangle}{a} - \frac{\langle\Pi\rangle}{a^2}, \\ v^\star & = & \overline{v} + \frac{\langle\Pi\rangle}{a}, & \Pi^\star & = & \overline{\Pi} + a\,\langle v\rangle\,, \\ Y^\star & = & \overline{Y} - \frac{\langle\Sigma\rangle}{b}, & \Sigma^\star & = & \overline{\Sigma} - b\,\langle Y\rangle\,. \end{array} \tag{3.50}$$

Assume that the parameter a is chosen large enough so that both $\tau^\star_{\mathbf{L}}$ and $\tau^\star_{\mathbf{R}}$ in (3.50) are positive (see for instance condition (3.51) below).

If $a > b$ then the eigenvalues are increasingly ordered as follows

$$\mu_1(\mathbf{v_L}) = (v - a\tau)_{\mathbf{L}} \leq (v - b\tau)(\mathbf{v}_1) \leq v(\mathbf{v}_2) \leq (v + b\tau)(\mathbf{v}_3) \leq (v + a\tau)(\mathbf{v}_4)$$

where the intermediate states are given by :

$$\mathbf{v}_1 = \begin{pmatrix} \rho^\star_{\mathbf{L}} \\ \rho^\star_{\mathbf{L}} v^\star \\ \rho^\star_{\mathbf{L}} \Pi^\star \\ \rho^\star_{\mathbf{L}} Y_{\mathbf{L}} \\ \rho^\star_{\mathbf{L}} \Sigma_{\mathbf{L}} \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} \rho^\star_{\mathbf{L}} \\ \rho^\star_{\mathbf{L}} v^\star \\ \rho^\star_{\mathbf{L}} \Pi^\star \\ \rho^\star_{\mathbf{L}} Y^\star \\ \rho^\star_{\mathbf{L}} \Sigma^\star \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} \rho^\star_{\mathbf{R}} \\ \rho^\star_{\mathbf{R}} v^\star \\ \rho^\star_{\mathbf{R}} \Pi^\star \\ \rho^\star_{\mathbf{R}} Y^\star \\ \rho^\star_{\mathbf{R}} \Sigma^\star \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} \rho^\star_{\mathbf{R}} \\ \rho^\star_{\mathbf{R}} v^\star \\ \rho^\star_{\mathbf{R}} \Pi^\star \\ \rho^\star_{\mathbf{R}} Y_{\mathbf{R}} \\ \rho^\star_{\mathbf{R}} \Sigma_{\mathbf{R}} \end{pmatrix}.$$

If $a < b$ then the eigenvalues are now increasingly ordered as follows

$$\mu_1(\mathbf{v_L}) = (v - b\tau)_{\mathbf{L}} \leq (v - a\tau)(\mathbf{v}_1) \leq v(\mathbf{v}_2) \leq (v + a\tau)(\mathbf{v}_3) \leq (v + b\tau)(\mathbf{v}_4)$$

where the intermediate states are given by :

$$\mathbf{v}_1 = \begin{pmatrix} \rho_{\mathbf{L}} \\ \rho_{\mathbf{L}} v_{\mathbf{L}} \\ \rho_{\mathbf{L}} \Pi_{\mathbf{L}} \\ \rho_{\mathbf{L}} Y^\star \\ \rho_{\mathbf{L}} \Sigma^\star \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} \rho^\star_{\mathbf{L}} \\ \rho^\star_{\mathbf{L}} v^\star \\ \rho^\star_{\mathbf{L}} \Pi^\star \\ \rho^\star_{\mathbf{L}} Y^\star \\ \rho^\star_{\mathbf{L}} \Sigma^\star \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} \rho^\star_{\mathbf{R}} \\ \rho^\star_{\mathbf{R}} v^\star \\ \rho^\star_{\mathbf{R}} \Pi^\star \\ \rho^\star_{\mathbf{R}} Y^\star \\ \rho^\star_{\mathbf{R}} \Sigma^\star \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} \rho_{\mathbf{R}} \\ \rho_{\mathbf{R}} v_{\mathbf{R}} \\ \rho_{\mathbf{R}} \Pi_{\mathbf{R}} \\ \rho_{\mathbf{R}} Y^\star \\ \rho_{\mathbf{R}} \Sigma^\star \end{pmatrix}.$$

Let us emphasize that the expressions stated in (3.50) are well defined as soon as the parameters a and b are strictly positive. Notice that by construction, the identities $\Pi_{L,R} = P(\mathbf{u}_{L,R})$ and $\Sigma_{L,R} = \sigma(\mathbf{u}_{L,R})$ hold true in (3.50). Furthermore, we point out that the requirement for positivity concerning the intermediate specific volumes $\tau^\star_{\mathbf{L}}$ and $\tau^\star_{\mathbf{R}}$ (e.g. a large enough) has nothing to do with the stability requirement (3.1.4) but just enforce for validity the proposed ordering of the eigenvalues (see the proof of Proposition 3.2.2). As a consequence, additional restrictions on the parameters a and b will have to be put : namely the satisfaction of the stability condition (3.1.4). This will be the matter of the next section. The reader will easily check that if a is chosen larger than $a_p(\mathbf{u_L}, \mathbf{u_R})$ where

$$a_p(\mathbf{u_L}, \mathbf{u_R}) := \frac{\langle v\rangle + \sqrt{\langle v\rangle^2 + 4\max(\tau_{\mathbf{L}}, \tau_{\mathbf{R}})|\langle\Pi\rangle|}}{2\min(\tau_{\mathbf{L}}, \tau_{\mathbf{R}})}, \tag{3.51}$$

then the intermediate state densities are positive.

PROOF As already underlined, the Riemann solution is uniquely made of contact discontinuities. The *j-th* one separates the states $\mathbf{v}_j$ and $\mathbf{v}_{j+1}$ and propagates with speed $\mu_j(\mathbf{v}_j) = \mu_j(\mathbf{v}_{j+1})$. At it

is well-known, the two sates $\mathbf{v}_j$ and $\mathbf{v}_{j+1}$ are not arbitrary but must solve the following Rankine-Hugoniot conditions :

$$-\mu_j(\mathbf{v}_j)(\mathbf{v}_{j+1}-\mathbf{v}_j)+(\mathbf{g}(\mathbf{v}_{j+1})-\mathbf{g}(\mathbf{v}_j))=0,\quad j=0,..,4. \tag{3.52}$$

Specializing this jump condition in the case of the density variable, we have $-\mu_j(\mathbf{v}_j)(\rho_{j+1}-\rho_j)+((\rho v)_{j+1}-(\rho v)_j)=0$, for each $j=0,..,4$ so that there exists a constant $\mathcal{M}_j$ satisfying

$$\mathcal{M}_j=(\mu_j(\mathbf{v}_j)-v_{j+1})\rho_{j+1}=(\mu_j(\mathbf{v}_j)-v_j)\rho_j. \tag{3.53}$$

Such a constant simply expresses the conservation of the mass flux across the *j-th* wave. But $\mu_j(\mathbf{v}_j)=v_j+c_j\tau_j$ with c_j equals either to $\pm a$, $\pm b$ or zero depending on the wave under consideration. A direct consequence of (3.53) is that $\mathcal{M}_j^2$ is equal either to a^2, b^2 or zero.

Next, consider the remaining jump conditions. These are easily seen to simplify thanks to (3.53) and we have

$$\left\{\begin{array}{l} -\mathcal{M}_j(v_{j+1}-v_j)+(\Pi_{j+1}-\Pi_j)=0,\\ -\mathcal{M}_j(\Pi_{j+1}-\Pi_j)+a^2(v_{j+1}-v_j)=0,\\ -\mathcal{M}_j(Y_{j+1}-Y_j)-(\Sigma_{j+1}-\Sigma_j)=0,\\ -\mathcal{M}_j(\Sigma_{j+1}-\Sigma_j)-b^2(Y_{j+1}-Y_j)=0. \end{array}\right. \tag{3.54}$$

Indeed, notice for instance that the associated jump condition in (ρ,v) rewrites $(\rho(v-\mu_j))_{j+1}\,v_{j+1}-(\rho(v-\mu_j))_j v_j+\Pi_{j+1}-\Pi_j=0$, which is nothing but the first identity in (3.54).

Next, let us observe that the first two relations in (3.54) give $(\mathcal{M}_j^2-a^2)(v_{j+1}-v_j)=0$, while the last two ones in (3.54) symmetrically yield $(\mathcal{M}_j^2-b^2)(Y_{j+1}-Y_j)=0$. We have therefore proved that across a discontinuity associated with the eigenvalue v, namely $\mathcal{M}_j^2=0$, necessarily all the variables stay constant except the density ρ which can achieve an arbitrary jump. Next, considering the discontinuity associated with the eigenvalues $v\pm a\tau$, e.g. $\mathcal{M}_j^2=a^2$, necessarily Y and Σ are continuous. Symmetrically, v and Π stay continuous across discontinuities associated with the eigenvalues $v\pm b\tau$.

To summarize, the intermediate states $\mathbf{v}_1$ to $\mathbf{v}_4$ can be made of at most one pair $(Y^\star,\Sigma^\star)$ distinct from $(Y_\mathbf{L},\Sigma_\mathbf{L})$ and $(Y_\mathbf{R},\Sigma_\mathbf{R})$. The pair $(Y^\star,\Sigma^\star)$ is solution (see indeed (3.52)) of $b(Y^\star-Y_\mathbf{L})-(\Sigma^\star-\Sigma_\mathbf{L})=0$ and $-b(Y_\mathbf{R}-Y^\star)-(\Sigma_\mathbf{R}-\Sigma^\star)=0$, which is exactly the definition of the required states in (3.50). For symmetric reasons, the intermediate states $\mathbf{v}_j$ can be made of at most a new velocity $v^\star$, a new pressure $\Pi^\star$ and two new densities distinct from $(\rho_\mathbf{L},v_\mathbf{L},\Pi_\mathbf{L})$ and $(\rho_\mathbf{R},v_\mathbf{R},\Pi_\mathbf{R})$. Such new states are defined when solving

$$a(v^\star-v_\mathbf{L})+(\Pi^\star-\Pi_\mathbf{L}) = 0, \tag{3.55}$$
$$-a(v_\mathbf{R}-v^\star)+(\Pi_\mathbf{R}-\Pi^\star) = 0. \tag{3.56}$$

These two relations are easily seen to yield $v^\star$ and $\Pi^\star$ in (3.50). Concerning the densities, it suffices to use the continuity of $\mathcal{M}_j$ across the associated wave to conclude. For instance, concerning the field associated with $v-a\tau$, we have $\rho_\mathbf{L}^\star(v^\star-(v_\mathbf{L}-a\tau_\mathbf{L}))=a$ so that $\tau_\mathbf{L}^\star=\tau_\mathbf{L}+\frac{v^\star-v_\mathbf{L}}{a}$. To conclude, let us establish the proposed ordering of the eigenvalues. Assume for instance that $a>b$ then the positivity of both $\tau_\mathbf{L}^\star$ and $\tau_\mathbf{R}^\star$ easily implies that

$$v_\mathbf{L}-a\tau_\mathbf{L}=v^\star-a\tau_\mathbf{L}^\star<v^\star-b\tau_\mathbf{L}^\star<v^\star<v^\star+b\tau_\mathbf{R}^\star<v^\star+a\tau_\mathbf{R}^\star=v_\mathbf{R}+a\tau_\mathbf{R}.$$

which completes the proof. □

3.2.2 Relaxation coefficients

In this section, we focus our attention on enforcing the stability requirements we have put forward in (3.18). As already stated, these stability conditions are fulfilled as soon as the pair (a, b) is chosen large enough but dealing with too large values of these parameters comes at the expense of a too large numerical dissipation. Indeed, the eigenvalues of the dissipative tensor $\mathcal{D}_{(a,b)}(\mathbf{u})$ introduced in Proposition 3.1.4 are seen to increase with a and b. Our motivation is therefore to lower the numerical dissipation when choosing optimal values of the pair (a, b). Such values will be chosen locally, namely for each Riemann problem at each cell interface $x_{i+1/2}$. Let us recall that this makes sense since the CFL condition (3.42) precludes interactions between two neighboring Riemann solutions (see (3.45)-(3.46) and (3.49)). Such a strategy will obviously allow us to locally adjust the numerical dissipation in an optimal way : this requirement is important because both the pressure law and the hydrodynamic one exhibit strongly varying derivatives. These directly affect the choice of a and b via the definition of $\mathcal{D}_{(a,b)}$.

Let two equilibrium states $\mathbf{u_L}$ and $\mathbf{u_R}$ be given defining the initial data $(\mathbf{v}(\mathbf{u_L}), \mathbf{v}(\mathbf{u_R}))$ for a Riemann problem. We propose to select $a(\mathbf{u_L}, \mathbf{u_R})$ and $b(\mathbf{u_L}, \mathbf{u_R})$ on the ground of Lemma 3.1.7 :

$$\gamma(\mathbf{u_L}, \mathbf{u_R}) = \max\left(\gamma(\mathbf{u_L}),\ \gamma(\mathbf{u_R})\right), \tag{3.57}$$

$$\widehat{a}_-(\mathbf{u_L}, \mathbf{u_R}) = \left(\sqrt{2}+1\right)\sqrt{\gamma(\mathbf{u_L}, \mathbf{u_R})}, \tag{3.58}$$

$$\widehat{b}_-(\mathbf{u_L}, \mathbf{u_R}) = \left(\sqrt{2}-1\right)\sqrt{\gamma(\mathbf{u_L}, \mathbf{u_R})}, \tag{3.59}$$

$$a_-(\mathbf{u_L}, \mathbf{u_R}) = \sqrt{\widehat{a}_-(\mathbf{u_L}, \mathbf{u_R}) + \max\left(A(\mathbf{u_L}),\ A(\mathbf{u_R})\right)}, \tag{3.60}$$

$$b_-(\mathbf{u_L}, \mathbf{u_R}) = \sqrt{\widehat{b}_-(\mathbf{u_L}, \mathbf{u_R}) + \max\left(B(\mathbf{u_L}),\ B(\mathbf{u_R})\right)}. \tag{3.61}$$

3.2.3 A maximum principle on mass fractions

The definition of the phase space Ω requires the mass fraction Y to stay within the invariant region $0 \leq Y \leq 1$. Such a property may nevertheless fail to be satisfied at the discrete level (see Larrouturou [40] in a distinct setting). Our motivation in this section is to enforce the validity of this maximum principle when seeking for appropriate pairs (a, b). More specifically, we exhibit a precise lower bound to be put locally (e.g. at each interface) on the parameter b. This additional desirable constraint will give a final definition for the optimal pair of parameters $(a^\star, b^\star)$. The first result of this section is as follows :

Theorem 3.2.3 *Let two equilibrium states $\mathbf{u_L}$ and $\mathbf{u_R}$ be given in Ω and define $\mathbf{v_L} = \mathbf{v}(\mathbf{u_L})$ and $\mathbf{v_R} = \mathbf{v}(\mathbf{u_R})$ according to (3.41). Assume that $a(\mathbf{u_L}, \mathbf{u_R}) > a_p(\mathbf{u_L}, \mathbf{u_R})$ (see (3.51)) and that $b(\mathbf{u_L}, \mathbf{u_R})$ satisfies*

$$b(\mathbf{u_L}, \mathbf{u_R}) \geq b_p(\mathbf{u_L}, \mathbf{u_R}) := \frac{|\rho_\mathbf{L}\Phi(\mathbf{u_L}) + \rho_\mathbf{R}\Phi(\mathbf{u_R})|}{2} + \frac{|\rho_\mathbf{L}\Phi(\mathbf{u_L}) - \rho_\mathbf{R}\Phi(\mathbf{u_R})|}{2}. \tag{3.62}$$

Then and with the definitions of the updated values (3.43)–(3.44),

$$\bar{Y}_\mathbf{L}(\mathbf{v_L}, \mathbf{v_R}) := \frac{\overline{\rho Y}_\mathbf{L}(\mathbf{v_L}, \mathbf{v_R})}{\bar{\rho}_\mathbf{L}(\mathbf{v_L}, \mathbf{v_R})} \quad \textit{and} \quad \bar{Y}_\mathbf{R}(\mathbf{v_L}, \mathbf{v_R}) := \frac{\overline{\rho Y}_\mathbf{R}(\mathbf{v_L}, \mathbf{v_R})}{\bar{\rho}_\mathbf{R}(\mathbf{v_L}, \mathbf{v_R})}$$

belongs to $[0, 1]$.

As an immediate consequence of Theorem 3.2.3, we have a

Corollary 3.2.4 *Assume that the parameters $a_{i+1/2}$ and $b_{i+1/2}$ are chosen respectively larger than $a_p(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n)$ and $b_p(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n)$ for all $i \in \mathbb{Z}$. Assume that at time level t^n, $Y_i^n \in [0,1]$ for all $i \in \mathbb{Z}$, then under the CFL condition (3.42), $Y_i^{n+1} \in [0,1]$ for all $i \in \mathbb{Z}$.*

PROOF OF THEOREM 3.2.3 Arguing about the structure of the Riemann solution, given in Proposition 3.2.2, $Y(w(\xi, \mathbf{v_L}, \mathbf{v_R}))$ may take at most three distinct values, namely :

$$Y_{\mathbf{L}}, \quad Y^\star = \frac{Y_{\mathbf{L}} + Y_{\mathbf{R}}}{2} + \frac{\Sigma_{\mathbf{R}} - \Sigma_{\mathbf{L}}}{2b}, \quad Y_{\mathbf{R}}, \tag{3.63}$$

where by construction, $\Sigma_{L,R} = \sigma(\mathbf{u}_{L,R})$ and by assumption, both $Y_{\mathbf{L}}$ and $Y_{\mathbf{R}}$ belong to $[0,1]$. We only have to prove that $Y^\star$ also remains in this range as soon as $b(\mathbf{u_L}, \mathbf{u_R}) \geq b_p$. Indeed, assuming such a property, we have by definition concerning the left averaged value :

$$\bar{Y}_{\mathbf{L}}(\mathbf{v_L}, \mathbf{v_R}) = \frac{2\Delta t}{\Delta x} \int_{-\frac{\Delta x}{2\Delta t}}^{0} Y(w(\xi, \mathbf{v_L}, \mathbf{v_R}))dm, \tag{3.64}$$

where the measure dm meets the definition

$$dm = \frac{\rho(w(\xi, \mathbf{v_L}, \mathbf{v_R}))\ d\xi}{\dfrac{2\Delta t}{\Delta x} \displaystyle\int_{-\frac{\Delta x}{2\Delta t}}^{0} \rho(w(\xi, \mathbf{v_L}, \mathbf{v_R}))\ d\xi}. \tag{3.65}$$

But since the parameter a is chosen larger than $a_p(\mathbf{u_L}, \mathbf{u_R})$, then Proposition 3.2.2 asserts the positivity of all the intermediate densities. As a consequence, the above measure dm is nothing but a probability measure, i.e., non negative with unit total mass on $[-\frac{\Delta x}{2\Delta t}, 0]$. Therefore, if $Y^\star$ belongs to $[0,1]$, then necessarily $\bar{Y}_{\mathbf{L}}(\mathbf{v_L}, \mathbf{v_R})$ shares the same property. Exactly the same steps apply to $\bar{Y}_{\mathbf{R}}(\mathbf{v_L}, \mathbf{v_R})$. So to carry out the proof, it is enough to show that the bound (3.62) implies the required property on $Y^\star$. From its formula definition given in (3.63), let us notice that such a property holds true as soon as the parameter $b(\mathbf{u_L}, \mathbf{u_R})$ obeys :

$$b \geq \max\left(-\frac{\sigma(\mathbf{u_L}) - \sigma(\mathbf{u_R})}{X_{\mathbf{L}} + X_{\mathbf{R}}},\ \frac{\sigma(\mathbf{u_L}) - \sigma(\mathbf{u_R})}{Y_{\mathbf{L}} + Y_{\mathbf{R}}} \right), \tag{3.66}$$

with $X = 1 - Y$ the liquid mass fraction. Let us now prove that (3.66) can be upper-bounded by $b_p(\mathbf{u_L}, \mathbf{u_R})$ defined in (3.62).

We will use the following lemma

Lemma 3.2.5 *Assume that $(r_\alpha, s_\alpha, t_\alpha) \in \mathbb{R}^3$ where $\alpha = R, L$ and satisfy :*

$$r_\alpha \in [0,1],\ s_\alpha \in [0,1],\ \overline{rs} \in [0, 1/2] \tag{3.67}$$

where we note $\overline{r} = (r_{\mathbf{L}} + r_{\mathbf{R}})/2$, $\quad \Delta r = r_{\mathbf{L}} - r_{\mathbf{R}}$. Then,

$$|\Delta(rst)| \leq |\bar{t}| + |\Delta t|/2. \tag{3.68}$$

PROOF OF LEMMA 3.2.5 The reader can easily check that :

$$\Delta(rst) = \Delta(rs)\ \bar{t}\ +\ \overline{rs}\ \Delta t. \tag{3.69}$$

But two the first hypotheses of (3.67) show that $(rs)_\alpha \in [0,1]$ and therefore $\Delta(rs) \in [0,1]$. The last hypothesis of (3.67) completes the proof. □

In a first step, according to $\sigma = \rho X Y \Phi$ with $X = 1 - Y$, we have :

$$\frac{\sigma(\mathbf{u_L}) - \sigma(\mathbf{u_R})}{Y_\mathbf{L} + Y_\mathbf{R}} = \frac{Y_\mathbf{L}}{Y_\mathbf{L} + Y_\mathbf{R}} \, X_\mathbf{L} \, \rho_\mathbf{L} \Phi(\mathbf{u_L}) - \frac{Y_\mathbf{R}}{Y_\mathbf{L} + Y_\mathbf{R}} \, X_\mathbf{R} \, \rho_\mathbf{R} \Phi(\mathbf{u_R})$$

Let us apply Lemma 3.2.5 with :

$$r_\alpha = Y_\alpha/(Y_\mathbf{L} + Y_\mathbf{R}) \quad s_\alpha = X_\alpha, \quad t_\alpha = \rho_\alpha \Phi(\mathbf{u}_\alpha), \quad \alpha = L, R. \tag{3.70}$$

It is clear that, whatever $Y_{L,R} \in [0,1]$, we have $\frac{Y_\alpha}{Y_\mathbf{L}+Y_\mathbf{R}} \in [0,1]$, with $\alpha = L, R$. Since it is obvious that $s_\alpha = X_\alpha \in [0,1]$, the two first hypotheses of the Lemma are true. In order to show the last one, let us note that : $Y_\mathbf{L} X_\mathbf{L} + Y_\mathbf{R} X_\mathbf{R} - (Y_\mathbf{L} + Y_\mathbf{R}) = -Y_\mathbf{L}^2 - Y_\mathbf{R}^2 \leq 0$ and therefore $0 \leq \frac{Y_\mathbf{L} X_\mathbf{L}}{Y_\mathbf{L}+Y_\mathbf{R}} + \frac{Y_\mathbf{R} X_\mathbf{R}}{Y_\mathbf{L}+Y_\mathbf{R}} \leq 1$ whatever $Y_{L,R} \in [0,1]$. The Lemma 3.2.5 implies therefore :

$$\left| \frac{\sigma(\mathbf{u_L}) - \sigma(\mathbf{u_R})}{Y_\mathbf{L} + Y_\mathbf{R}} \right| \leq b_p(\mathbf{u_L}, \mathbf{u_R}). \tag{3.71}$$

In a second step, we use the same arguments in order to bound :

$$\frac{\sigma(\mathbf{u_L}) - \sigma(\mathbf{u_R})}{X_\mathbf{L} + X_\mathbf{R}} = \frac{X_\mathbf{L}}{X_\mathbf{L} + X_\mathbf{R}} \, Y_\mathbf{L} \, \rho_\mathbf{L} \Phi(\mathbf{u_L}) - \frac{X_\mathbf{R}}{X_\mathbf{L} + X_\mathbf{R}} \, Y_\mathbf{R} \, \rho_\mathbf{R} \Phi(\mathbf{u_R}).$$

Indeed, we now use Lemma 3.2.5 with :

$$r_\alpha = X_\alpha/(X_\mathbf{L} + X_\mathbf{R}), \quad s_\alpha = Y_\alpha, \quad t_\alpha = \rho_\alpha \Phi(\mathbf{u}_\alpha), \quad \alpha = L, R. \tag{3.72}$$

It is clear that, whatever $X_{L,R} \in [0,1]$, we have $\frac{X_\alpha}{X_\mathbf{L}+X_\mathbf{R}} \in [0,1]$, with $\alpha = L, R$. In order to check the last hypothesis of the Lemma, let us note that $X_\mathbf{L} Y_\mathbf{L} + X_\mathbf{R} Y_\mathbf{R} - (X_\mathbf{L} + X_\mathbf{R}) = -X_\mathbf{L}^2 - X_\mathbf{R}^2 \leq 0$ and therefore $\frac{X_\mathbf{L} Y_\mathbf{L}}{X_\mathbf{L}+X_\mathbf{R}} + \frac{X_\mathbf{R} Y_\mathbf{R}}{X_\mathbf{L}+X_\mathbf{R}} \in [0,1]$ whatever $Y_{L,R} \in [0,1]$. The Lemma 3.2.5 finally implies :

$$\left| \frac{\sigma(\mathbf{u_L}) - \sigma(\mathbf{u_R})}{X_\mathbf{L} + X_\mathbf{R}} \right| \leq b_p(\mathbf{u_L}, \mathbf{u_R}), \tag{3.73}$$

which concludes the proof □.

We conclude the present section by establishing Corollary 3.2.4.

PROOF Considering the updating formula (3.47)–(3.48), we clearly have

$$Y_i^{n+1} := \frac{(\rho Y)_i^{n+1}}{\rho_i^{n+1}} = \frac{\overline{(\rho Y)}_\mathbf{R}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n) + \overline{(\rho Y)}_\mathbf{L}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n)}{\overline{\rho}_\mathbf{R}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n) + \overline{\rho}_\mathbf{L}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n)}, \tag{3.74}$$

where for instance and obviously $\overline{(\rho Y)}_\mathbf{L}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) = \overline{\rho}_\mathbf{L}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) \times \overline{Y}_\mathbf{L}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n)$. Since the parameter $a_{i+1/2}$ is larger than $a_p(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n)$ at each interface $x_{i+1/2}$, we necessarily have

$$\overline{\rho}_\mathbf{R}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n) > 0 \quad \text{and} \quad \overline{\rho}_\mathbf{L}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) > 0.$$

(see indeed the formula (3.43) and (3.44) when arguing about the positivity of the intermediate densities.) As a direct consequence, formula (3.74) is nothing but a convex combination of $\bar{Y}_\mathbf{L}$ and $\bar{Y}_\mathbf{R}$ which we know to belong to $[0,1]$ in view of Theorem 3.2.3 since the $b_{i+1/2}$ are chosen larger than $b_p(\mathbf{u}_i^n, \mathbf{u}_{i+1}^n)$ for all $i \in \mathbb{Z}$. □

3.2.4 On the optimal choice of the pair (a, b)

In this section, we propose a definitive version of the optimal pair $(a^\star, b^\star)$ to be associated with a given local Riemann problem. These relaxation coefficients are defined by

$$a^\star(\mathbf{u_L}, \mathbf{u_R}) = \max\left[a_-\left(\mathbf{u_L}, \mathbf{u_R}\right), a_p\left(\mathbf{u_L}, \mathbf{u_R}\right)\right], \tag{3.75}$$

$$b^\star(\mathbf{u_L}, \mathbf{u_R}) = \max\left[b_-\left(\mathbf{u_L}, \mathbf{u_R}\right), b_p\left(\mathbf{u_L}, \mathbf{u_R}\right)\right]. \tag{3.76}$$

This choice guarantees that the stability requirement (3.18), the positivity of the intermediate densities and the maximum principle for the mass fraction $Y^\star$ in (3.50) are valid.

Remark 3.2.6 *The relaxation coefficients are designed in order to ensure :*

1. *the stability of the first order asymptotic equilibrium system (3.16) thanks to the Chapman-Enskog expansion,*
2. *physical properties of the approximate solution (positivity of the density and positivity of the mass fractions).*

From a theoretical point of view, the weakness of the first point (i.e. the Chapman-Enskog analysis) is that is is just formal and heuristical and does not always give the right condition. However, the second point (the physical properties satisfied by the approximate solution) ensures that the relaxation scheme is robust. Moreover, from a pratical point of view, the scheme is stable enough to handle difficult cases, which we are going to verify.

3.3 Numerical results

The relaxation scheme is now applied to a few Riemann problems over a domain of 100 m, the discontinuity between the two initial constant states being located at $x = 50$. In the benchmarks presented, we use a uniform mesh with $\Delta x = 0.5$ m and a CFL ratio equal to 0.5.

The liquid phase is assumed to be incompressible and we take $\rho_{\mathrm{L}} = 1000$ kg/m^3 with a constant sound speed a_{G} in the gas phase. Such considerations lead to the pressure law

$$p(\rho, Y) = a_{\mathrm{G}}^2 \frac{\rho Y}{R_{\mathrm{G}}(\mathbf{u})}, \tag{3.77}$$

where $R_{\mathrm{G}}(\mathbf{u}) = 1 - R_{\mathrm{L}}(\mathbf{u})$ with $R_{\mathrm{L}}(\mathbf{u}) = \rho(1-Y)/\rho_{\mathrm{L}}$. In Experiments 1–2, we work with a no-slip closure law, in Experiment 3, the hydrodynamic law is that of Zuber-Findlay, in Experiment 4, we use a dispersed slip law and in Experiment 5, we use a synthetic slip law.

3.3.1 No-slip law

By "no-slip law", we mean that $\Phi \equiv 0$. The conjunction of the pressure law (3.77) and the no-slip law gives rise to the hyperbolicity of the original system (3.5), which now reads

$$\left\{\begin{array}{lcll} \partial_t(\rho) & + & \partial_x(\rho v) & = 0, \\ \partial_t(\rho v) & + & \partial_x(\rho v^2 + p) & = s, \\ \partial_t(\rho Y) & + & \partial_x(\rho Y v) & = 0. \end{array}\right. \tag{3.78}$$

As before, we assume $s \equiv 0$. The eigenvalues and the Riemann invariants of (3.78) are summarized in the following Table.

Field	Eigenvalue	Weak invariant	Strong invariant
λ^-	$v-c$	Y, ϕ^-	
λ^0	v	p, v	Y
λ^+	$v+c$	Y, ϕ^+	

Here, we note

$$c = \frac{a_{\mathrm{G}}\sqrt{Y}}{1-\dfrac{\rho(1-Y)}{\rho_{\mathrm{L}}}}, \qquad \phi^{\pm} = \left[\frac{1}{\rho} - \frac{1-Y}{\rho_{\mathrm{L}}}\right] \exp\left(\frac{\pm v}{a_{\mathrm{G}}\sqrt{Y}}\right).$$

The gas mass fraction Y, being a strong invariant for λ_0, is therefore a weak invariant for λ^+ and λ^-. Thus, Y is constant across a rarefaction wave associated to $\lambda^{\pm}$. It is also easy to show that Y is constant across a $\lambda^{\pm}$-shock, the Hugoniot locus of which is given by $[Y]=0$ and $[v]^2+[p]\,[\tau]=0$.

Entropic conditions are also easy to take into account, and will not be detailed here. The main idea of this part is to show that, in the no-slip configuration, it is possible to compute the solution to the Riemann problem analytically. This allows us to design the following simple test cases.

3.3.1.1 Experiment 1

Consider the left (L) and right (R) states

$$\begin{pmatrix}\rho\\ Y\\ v\end{pmatrix}_L = \begin{pmatrix}500\\ 0.2\\ 34.4233\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\rho\\ Y\\ v\end{pmatrix}_R = \begin{pmatrix}400\\ 0.2\\ 50\end{pmatrix}.$$

These have been tailored so that the solution to the Riemann problem is a pure λ^--rarefaction. With $a_{\mathrm{G}} = 100$ m/s, the speeds of propagation of the fronts are $\lambda^-_{\mathbf{L}} = -40.12$ m/s and $\lambda^-_{\mathbf{R}} = -15.77$ m/s. Snapshots in Fig. 3.2 correspond to time $T = 0.8$ s.

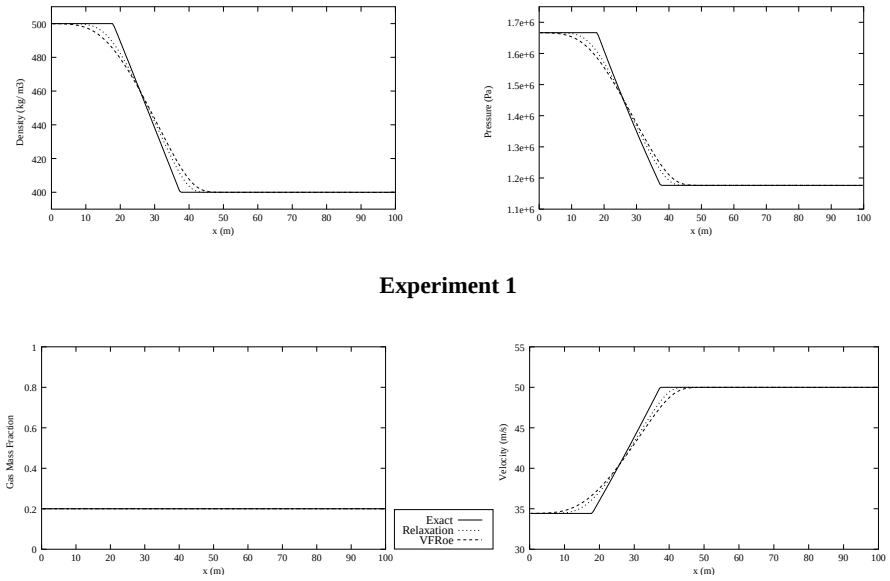

FIG. 3.2 – Experiment 1—A λ^--rarefaction for the no-slip closure law.

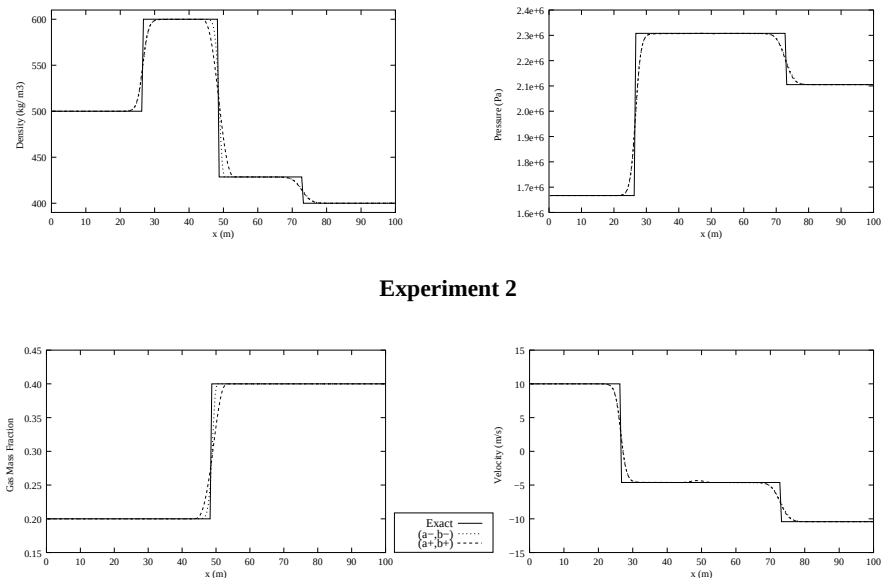

FIG. 3.3 – Experiment 2—A 3-wave Riemann problem for the no-slip closure law.

3.3.1.2 Experiment 2

Consider the two states

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 500 \\ 0.2 \\ 10 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 400 \\ 0.4 \\ -10.4261 \end{pmatrix}.$$

The theoretical solution to the Riemann problem is now a Shock–Contact Discontinuity–Shock pattern propagating at speeds $\sigma^- = -77.7$ m/s, $\sigma^0 = -4.62$ m/s and $\sigma^+ = 76.7$ m/s Snapshots in Fig. 3.3 correspond to time $T = 0.3$ s.

For this run, there is a comparison between the two possible choices for the relaxation coefficients : (a^+, b^+) and (a^-, b^-). It is clearly seen that the former pair leads to a less accurate contact discontinuity.

3.3.2 Zuber-Findlay law

Originally, the Zuber-Findlay hydrodynamic law [71] is an experimental observation that holds for intermittent flows, stating that[1] $v_G = c_0(R_G v_G + R_L v_L) + c_1$, valid for $c_0 > 1$ and $R_G < 1/c_0$. This equation can be expressed in terms of velocity slip by

$$\Phi(\mathbf{u}) = \frac{(c_0 - 1)v + c_1}{c_0[(1-Y)R_G(\mathbf{u}) - Y R_L(\mathbf{u})] - (1-Y)}.$$

The Zuber-Findlay law has been extensively studied by Benzoni-Gavage [11] in her thesis. Benzoni succeeded in constructing solutions to some Riemann problems involving this law. Inspired from this work, we consider

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 453.197 \\ 0.00705 \\ 24.8074 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 454.915 \\ 0.0108 \\ 1.7461 \end{pmatrix}.$$

[1] We recall that R_L is the area liquid fraction and $R_G = 1 - R_L$. Whenever the liquid is incompressible, we have $R_L = \rho(1-Y)/\rho_L$

The other parameters are : $a_{\mathrm{G}} = 300$ m/s, $c_0 = 1.07$, $c_1 = 0.2162$ m/s. The solution has the pattern Shock–Contact Discontinuity–Shock propagating at speeds $\sigma^- = -40.03$ m/s, $\sigma^0 = 10$ m/s and $\sigma^+ = 67.24$ m/s. Snapshots in Fig. 3.4 correspond to time $T = 0.5$ s.

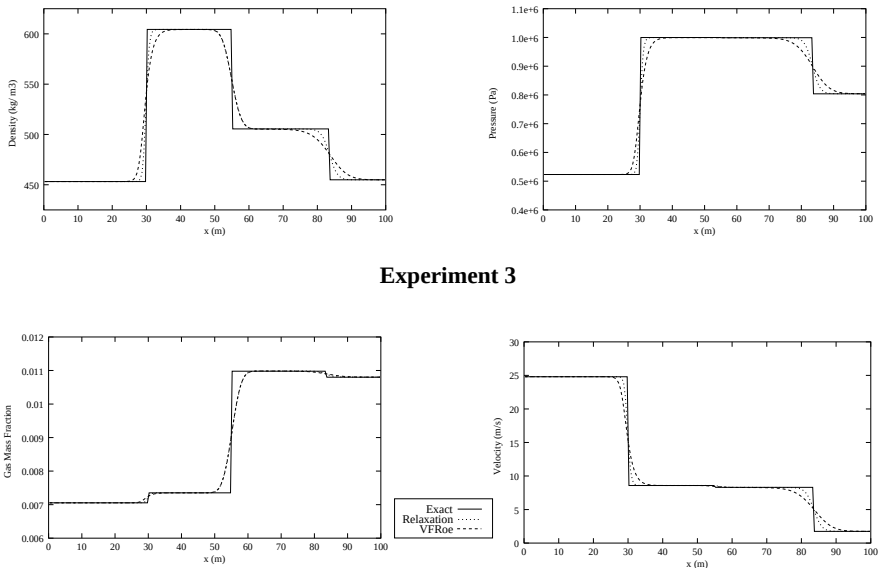

FIG. 3.4 – Experiment 3—A 3-wave Riemann problem for the Zuber-Findlay law.

3.3.3 Dispersed law

We now use an hydrodynamic law that holds for two-phase flows with little bubbles of gas :

$$\Phi(\mathbf{u}) = -\delta / R_{\mathrm{L}}(\mathbf{u})$$

where $\delta = 1.53 \sqrt[4]{g\sigma/\rho_{\mathrm{L}}} \sin\theta$ and $\sigma = 7.5.10^{-5}$ is the superficial stress. Inspired from the work of Benzoni-Gavage [11], we consider

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 901.11 \\ 1.2330.10^{-3} \\ 0.95027 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 208.88 \\ 4.2552.10^{-2} \\ 0.78548 \end{pmatrix}.$$

The other parameters are : $a_{\mathrm{G}} = 300\ m/s$, $\theta = \frac{\pi}{2}$ and $g = 9.81\ m/s^2$. The solution is a Contact Discontinuity propagating at speed $\sigma = 1\ m/s$. Snapshots in Fig. 3.5 correspond to time $T = 20\ s$.

3.3.4 Synthetic law

When dealing with industrial cases, we must use a hydrodynamic law which can handle all types of flows : stratified, intermittent (the Zuber-Findlay law), dispersed, etc... However, each slip-law $\Phi(\mathbf{u})$ is only valid for certain values of $\mathbf{u}$. In practice, physicists draw maps which describes the subsets of the admissible space in which a determined slip-law is valid. It happens that from one subset to another subset, the slip-law is non-continuous ! An attempt has been made to design a continuous slip-law $\Phi(\mathbf{u})$ that is able to represent various flow regimes. We call it the "synthetic law". Here, we will use the variable $\mathbf{w} = \mathcal{L}(\mathbf{u}) = (R_{\mathrm{G}}, p, u_s)$ where $u_s = R_{\mathrm{L}} v_{\mathrm{L}} + R_{\mathrm{G}} v_{\mathrm{G}}$ is the superficial velocity. The function $\mathcal{L}$ is a non-linear operator which will not be detailed here. We will describe in the next paragraph how we design the function $\Psi(\mathbf{w})$, from which we can recover the slip function

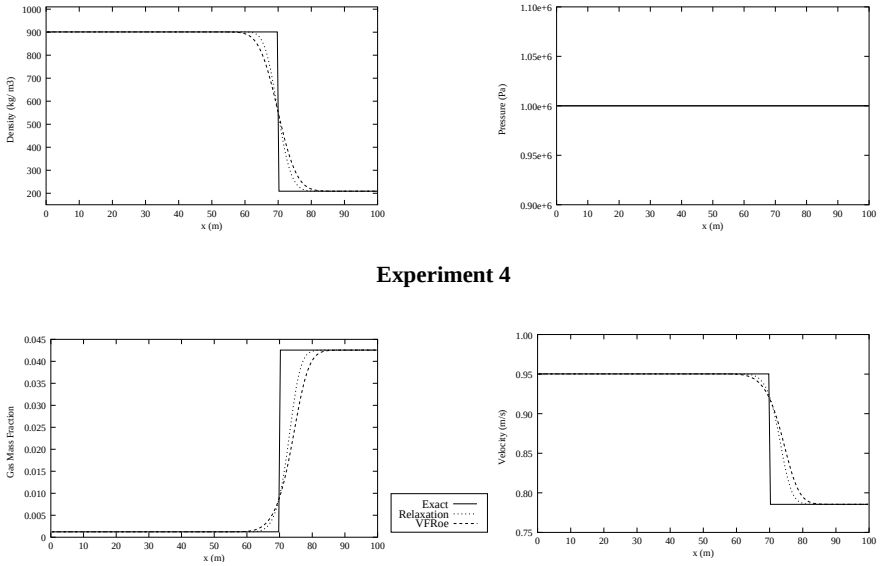

FIG. 3.5 – Experiment 4—Contact Discontinuity for the dispersed law.

$\Psi(\mathbf{u})$ by the change of variable $\Phi(\mathbf{u}) = \mathcal{M}(\Psi(\mathbf{w}), \mathbf{w})$ with $\mathbf{w} = \mathcal{L}(\mathbf{u})$. Again, the non-linear operator $\mathcal{M}$ will not be detailed here. We assume that there are just 3 types of flows : the dispersed flow with $0 \leq R_{\mathrm{G}} < 1/3$, the Zuber-Findlay flow with $1/3 \leq R_{\mathrm{G}} < 2/3$ and the stratified flow with $2/3 \leq R_{\mathrm{G}} \leq 1$. We then write the slip law under the form :

$$U_{\mathrm{G}} = \Psi(\mathbf{w}) = u_s \Psi_0(R_{\mathrm{G}}, p) + \Psi_1(R_{\mathrm{G}}, p) \tag{3.79}$$

where $U_{\mathrm{G}} = R_{\mathrm{G}} v_{\mathrm{G}}$. The function Ψ_0 and Ψ_1 are then computed by polynomial interpolation in order to ensure the continuity of Ψ, the validity of the law on the different ranges and compatibility conditions.

The previous lines show that **the function $\Phi(\mathbf{u})$ is highly non-linear**. In this case, the solution of the Riemann problem is only composed of shocks since all the fields are really non-linear. No analytical solution is known at this date for this slip-law. However, we can compute states which are connected by the two Rankine-Hugoniot conditions $[v]^2 + [P][\tau] = 0$ and $[v][Y] - [\sigma][\tau] = 0$, which are solved by a non-linear solver. The two states

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 492.7 \\ 0.20763 \\ -50.331 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 400. \\ 0.2 \\ -65.65 \end{pmatrix}$$

satisfy these conditions. The eigenvalues of the Jacobian matrices are $\lambda_{\mathbf{L}} = (-124.9,\ -55.0,\ 26.0)$ and $\lambda_{\mathbf{R}} = (-132.0,\ -73.6,\ 2.4)$. The speed of the discontinuity is $s = 15.766$ (m/s). Therefore, the Lax entropy condition shows that the exact solution is a 3-shock since $\lambda^3_{\mathbf{L}} = 26.0 > s > \lambda^3_{\mathbf{R}} = 2.4$. The other parameters are : $a_{\mathrm{G}} = 100\ m/s$ and $\theta = 0$. Snapshots in Fig. 3.6 correspond to time $T = 2\ s$.

Conclusion

In view of the numerical results, this first attempt to solve a simplified two-phase flow system by a relaxation method can be considered as successful. This is all the more astonishing that, at first

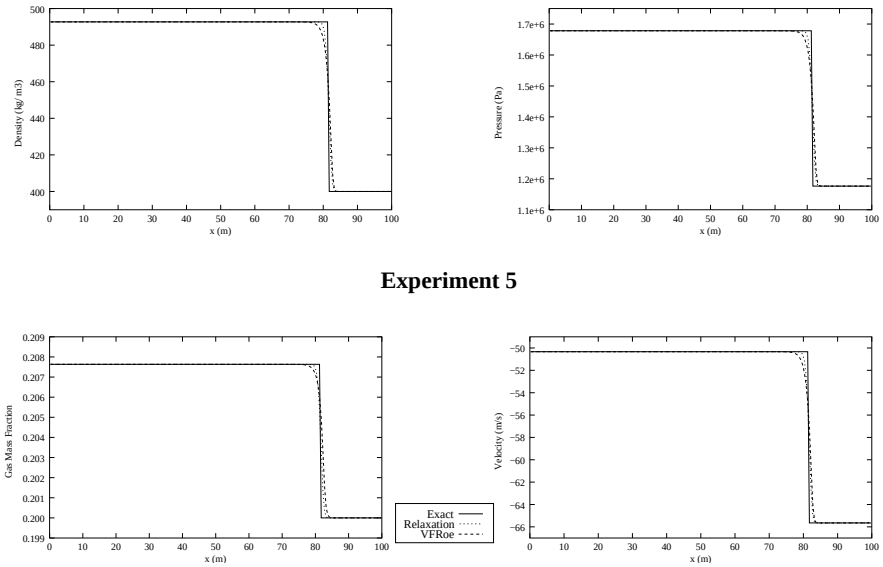

FIG. 3.6 – Experiment 5—3-shock for the synthetic law.

sight, the relaxation procedure may seem "brute", in comparison with the usual relaxation strategies for other problems [18].

The next steps are : incorporation of source terms, second-order schemes and boundary conditions. One important step will be the extension of this explicit scheme to an linearly implicit scheme, in order to get rid of the CFL time step limitation. Variable sections and multicomponent flows can also be envisaged.

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