FULL-SPACE IMPEDANCE LAPLACE PROBLEM

D.1 Introduction

In this appendix we study the perturbed full-space or free-space impedance Laplace problem, also known as the exterior impedance Laplace problem in 3D, using integral equation techniques and the boundary element method.

We consider the problem of the Laplace equation in three dimensions on the exterior of a bounded obstacle with an impedance boundary condition. The perturbed full-space impedance Laplace problem is not strictly speaking a wave scattering problem, but it can be regarded as a limit case of such a problem when the frequency tends towards zero (vid. Appendix E). It can be also regarded as a surface wave problem around a bounded three-dimensional obstacle. The two-dimensional problem has been already treated thoroughly in Appendix B.

For the problem treated herein we follow mainly Nédélec (1977, 1979, 2001) and Raviart (1991). Further related books and doctorate theses are Chen & Zhou (1992), Evans (1998), Giroire (1987), Hsiao & Wendland (2008), Johnson (1987), Kellogg (1929), Kress (1989), Rjasanow & Steinbach (2007), and Steinbach (2008). Some articles that deal specifically with the Laplace equation with an impedance boundary condition are Ahner & Wiener (1991), Lanzani & Shen (2004), and Medková (1998). The mixed boundary-value problem is treated by Wendland, Stephan & Hsiao (1979). Interesting theoretical details on transmission problems can be found in Costabel & Stephan (1985). The boundary element calculations can be found in Bendali & Devys (1986). The use of cracked domains is studied by Medková & Krutitskii (2005), and the inverse problem by Fasino & Inglese (1999) and Lin & Fang (2005). Applications of the Laplace problem can be found, among others, for electrostatics (Jackson 1999), for conductivity in biomedical imaging (Ammari 2008), and for incompressible three-dimensional potential flows (Spurk 1997).

The Laplace equation does not allow the propagation of volume waves inside the considered domain, but the addition of an impedance boundary condition permits the propagation of surface waves along the boundary of the obstacle. The main difficulty in the numerical treatment and resolution of our problem is the fact that the exterior domain is unbounded. We solve it therefore with integral equation techniques and the boundary element method, which require the knowledge of the Green's function.

This appendix is structured in 13 sections, including this introduction. The differential problem of the Laplace equation in a three-dimensional exterior domain with an impedance boundary condition is presented in Section D.2. The Green's function and its far-field expression are computed respectively in Sections D.3 and D.4. Extending the differential problem towards a transmission problem, as done in Section D.5, allows its resolution by using integral equation techniques, which is discussed in Section D.6. These techniques allow also to represent the far field of the solution, as shown in Section D.7. A particular problem that takes as domain the exterior of a sphere is solved analytically in Section D.8. The appropriate function spaces and some existence and uniqueness results for the solution of the problem are presented in Section D.9. By means of the variational formulation developed in Section D.10, the obtained integral equation is discretized using the boundary element method, which is described in Section D.11. The boundary element calculations required to build the matrix of the linear system resulting from the numerical discretization are explained in Section D.12. Finally, in Section D.13 a benchmark problem based on the exterior sphere problem is solved numerically.

D.2 Direct perturbation problem

We consider an exterior open and connected domain $\Omega_e \subset \mathbb{R}^3$ that lies outside a bounded obstacle Ω_i and whose boundary $\Gamma = \partial\Omega_e = \partial\Omega_i$ is regular (e.g., of class C^2), as shown in Figure D.1. As a perturbation problem, we decompose the total field u_T as $u_T = u_W + u$, where u_W represents the known field without obstacle, and where u denotes the perturbed field due its presence, which has bounded energy. The direct perturbation problem of interest is to find the perturbed field u that satisfies the Laplace equation in Ω_e, an impedance boundary condition on Γ, and a decaying condition at infinity. We consider that the origin is located in Ω_i and that the unit normal $\boldsymbol{n}$ is taken always outwardly oriented of Ω_e, i.e., pointing inwards of Ω_i.

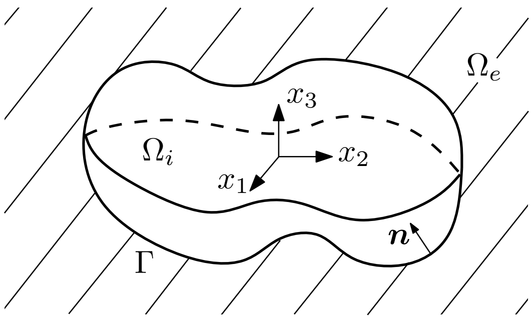

FIGURE D.1. Perturbed full-space impedance Laplace problem domain.

The total field u_T satisfies the Laplace equation

$$\Delta u_T = 0 \qquad \text{in } \Omega_e, \tag{D.1}$$

which is also satisfied by the fields u_W and u, due linearity. For the perturbed field u we take also the inhomogeneous impedance boundary condition

$$-\frac{\partial u}{\partial n} + Zu = f_z \qquad \text{on } \Gamma, \tag{D.2}$$

where Z is the impedance on the boundary, and where the impedance data function f_z is assumed to be known. If $Z = 0$ or $Z = \infty$, then we retrieve respectively the classical Neumann or Dirichlet boundary conditions. In general, we consider a complex-valued impedance $Z(\boldsymbol{x})$ depending on the position $\boldsymbol{x}$. The function $f_z(\boldsymbol{x})$ may depend on Z and u_w, but is independent of u.

The Laplace equation (D.1) admits different kinds of non-trivial solutions u_W, when we consider the domain Ω_e as the unperturbed full-space $\mathbb{R}^3$. One kind of solutions are the harmonic polynomials in $\mathbb{R}^3$. There exist likewise other harmonic non-polynomial functions that satisfy the Laplace equation in $\mathbb{R}^3$, but which have a bigger growth at infinity than any polynomial, e.g., the exponential functions

$$u_W(\boldsymbol{x}) = e^{\boldsymbol{a}\cdot\boldsymbol{x}}, \qquad \text{where } \boldsymbol{a} \in \mathbb{C}^3 \text{ and } a_1^2 + a_2^2 + a_3^2 = 0. \tag{D.3}$$

Any such function can be taken as the known field without perturbation u_W, which holds in particular for all the constant and linear functions in $\mathbb{R}^3$.

For the perturbed field u in the exterior domain Ω_e, though, these functions represent undesired non-physical solutions, which have to be avoided in order to ensure uniqueness of the solution u. To eliminate them, it suffices to impose for u an asymptotic decaying behavior at infinity that excludes the polynomials. This decaying condition involves finite energy throughout Ω_e and can be interpreted as an additional boundary condition at infinity. In our case it is given, for a great value of $|\boldsymbol{x}|$, by

$$u(\boldsymbol{x}) = \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{and} \qquad |\nabla u(\boldsymbol{x})| = \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^2}\right). \tag{D.4}$$

It can be expressed equivalently, for some constants $C > 0$, by

$$|u(\boldsymbol{x})| \leq \frac{C}{|\boldsymbol{x}|} \qquad \text{and} \qquad |\nabla u(\boldsymbol{x})| \leq \frac{C}{|\boldsymbol{x}|^2} \qquad \text{as } |\boldsymbol{x}| \to \infty. \tag{D.5}$$

In fact, the decaying condition can be even stated as

$$u(\boldsymbol{x}) = \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^\alpha}\right) \qquad \text{and} \qquad |\nabla u(\boldsymbol{x})| = \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^{1+\alpha}}\right) \qquad \text{for } 0 < \alpha \leq 1, \tag{D.6}$$

or as the more weaker and general formulation

$$\lim_{R\to\infty} \int_{S_R} \frac{|u|^2}{R^2}\,\mathrm{d}\gamma = 0 \qquad \text{and} \qquad \lim_{R\to\infty} \int_{S_R} |\nabla u|^2\,\mathrm{d}\gamma = 0, \tag{D.7}$$

where $S_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$ is the sphere of radius R and where the boundary differential element in spherical coordinates is given by $\mathrm{d}\gamma = R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

The perturbed full-space impedance Laplace problem can be finally stated as

$$\begin{cases} \text{Find } u : \Omega_e \to \mathbb{C} \text{ such that} \\ \Delta u = 0 & \text{in } \Omega_e, \\ -\dfrac{\partial u}{\partial n} + Zu = f_z & \text{on } \Gamma, \\ |u(\boldsymbol{x})| \leq \dfrac{C}{|\boldsymbol{x}|} & \text{as } |\boldsymbol{x}| \to \infty, \\ |\nabla u(\boldsymbol{x})| \leq \dfrac{C}{|\boldsymbol{x}|^2} & \text{as } |\boldsymbol{x}| \to \infty. \end{cases} \tag{D.8}$$

D.3 Green's function

The Green's function represents the response of the unperturbed system (without an obstacle) to a Dirac mass. It corresponds to a function G, which depends on a fixed source point $\boldsymbol{x} \in \mathbb{R}^3$ and an observation point $\boldsymbol{y} \in \mathbb{R}^3$. The Green's function is computed in the sense of distributions for the variable $\boldsymbol{y}$ in the full-space $\mathbb{R}^3$ by placing at the right-hand side of the Laplace equation a Dirac mass $\delta_{\boldsymbol{x}}$, centered at the point $\boldsymbol{x}$. It is therefore a solution $G(\boldsymbol{x},\cdot) : \mathbb{R}^3 \to \mathbb{C}$ for the radiation problem of a point source, namely

$$\Delta_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \qquad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{D.9}$$

Due to the radial symmetry of the problem (D.9), it is natural to look for solutions in the form $G = G(r)$, where $r = |\boldsymbol{y} - \boldsymbol{x}|$. By considering only the radial component, the Laplace equation in $\mathbb{R}^3$ becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}G}{\mathrm{d}r}\right) = 0, \qquad r > 0. \tag{D.10}$$

The general solution of (D.10) is of the form

$$G(r) = \frac{C_1}{r} + C_2, \tag{D.11}$$

for some constants C_1 and C_2. The choice of C_2 is arbitrary, while C_1 is fixed by the presence of the Dirac mass in (D.9). To determine C_1, we have to perform thus a computation in the sense of distributions (cf. Gel'fand & Shilov 1964), using the fact that G is harmonic for $r \neq 0$. For a test function $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we have by definition that

$$\langle \Delta_{\boldsymbol{y}} G, \varphi \rangle = \langle G, \Delta\varphi \rangle = \int_{\mathbb{R}^3} G\,\Delta\varphi\,\mathrm{d}\boldsymbol{y} = \lim_{\varepsilon\to 0}\int_{r\geq\varepsilon} G\,\Delta\varphi\,\mathrm{d}\boldsymbol{y}. \tag{D.12}$$

We apply here Green's second integral theorem (A.613), choosing as bounded domain the spherical shell $\varepsilon \leq r \leq a$, where a is large enough so that the test function $\varphi(\boldsymbol{y})$, of bounded support, vanishes identically for $r \geq a$. Then

$$\int_{r\geq\varepsilon} G\,\Delta\varphi\,\mathrm{d}\boldsymbol{y} = \int_{r\geq\varepsilon} \Delta_{\boldsymbol{y}} G\,\varphi\,\mathrm{d}\boldsymbol{y} - \int_{r=\varepsilon} G\frac{\partial\varphi}{\partial r}\,\mathrm{d}\gamma + \int_{r=\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}\varphi\,\mathrm{d}\gamma, \tag{D.13}$$

where $\mathrm{d}\gamma$ is the line element on the sphere $r = \varepsilon$. Now

$$\int_{r\geq\varepsilon} \Delta_{\boldsymbol{y}} G\,\varphi\,\mathrm{d}\boldsymbol{y} = 0, \tag{D.14}$$

since outside the ball $r \leq \varepsilon$ the function G is harmonic. As for the other terms, by replacing (D.11), we obtain that

$$\int_{r=\varepsilon} G\frac{\partial\varphi}{\partial r}\,\mathrm{d}\gamma = \left(\frac{C_1}{\varepsilon} + C_2\right)\int_{r=\varepsilon}\frac{\partial\varphi}{\partial r}\,\mathrm{d}\gamma = \mathcal{O}(\varepsilon), \tag{D.15}$$

and

$$\int_{r=\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}\varphi\,\mathrm{d}\gamma = -\frac{C_1}{\varepsilon^2}\int_{r=\varepsilon}\varphi\,\mathrm{d}\gamma = -4\pi C_1 S_\varepsilon(\varphi), \tag{D.16}$$

where $S_\varepsilon(\varphi)$ is the mean value of $\varphi(\boldsymbol{y})$ on the sphere of radius ε and centered at $\boldsymbol{x}$. In the limit as $\varepsilon \to 0$, we obtain that $S_\varepsilon(\varphi) \to \varphi(\boldsymbol{x})$, so that

$$\langle \Delta_{\boldsymbol{y}} G, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{r \geq \varepsilon} G \, \Delta\varphi \, \mathrm{d}\boldsymbol{y} = -4\pi C_1 \varphi(\boldsymbol{x}) = -4\pi C_1 \langle \delta_{\boldsymbol{x}}, \varphi \rangle. \tag{D.17}$$

Thus if $C_1 = -1/4\pi$, then (D.9) is fulfilled. When we consider not only radial solutions, then the general solution of (D.9) is given by

$$G(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\pi|\boldsymbol{y} - \boldsymbol{x}|} + \phi(\boldsymbol{x}, \boldsymbol{y}), \tag{D.18}$$

where $\phi(\boldsymbol{x}, \boldsymbol{y})$ is any harmonic function in the variable $\boldsymbol{y}$, i.e., such that $\Delta_{\boldsymbol{y}}\phi = 0$ in $\mathbb{R}^3$, e.g., an harmonic polynomial in $\mathbb{R}^3$ or a function of the form of (D.3).

If we impose additionally, for a fixed $\boldsymbol{x}$, the asymptotic decaying condition

$$|\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})| = \mathcal{O}\left(\frac{1}{|\boldsymbol{y}|^2}\right) \qquad \text{as} \;\; |\boldsymbol{y}| \longrightarrow \infty, \tag{D.19}$$

then we eliminate any polynomial (or bigger) growth at infinity, including constant and logarithmic growth. The Green's function satisfying (D.9) and (D.19) is finally given by

$$G(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\pi|\boldsymbol{y} - \boldsymbol{x}|}, \tag{D.20}$$

being its gradient

$$\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{y} - \boldsymbol{x}}{4\pi|\boldsymbol{y} - \boldsymbol{x}|^3}. \tag{D.21}$$

We can likewise define a gradient with respect to the $\boldsymbol{x}$ variable by

$$\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{x} - \boldsymbol{y}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3}, \tag{D.22}$$

and a double-gradient matrix by

$$\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = -\frac{\boldsymbol{I}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3} + \frac{3(\boldsymbol{x} - \boldsymbol{y}) \otimes (\boldsymbol{x} - \boldsymbol{y})}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^5}, \tag{D.23}$$

where $\boldsymbol{I}$ denotes a 3×3 identity matrix and where $\otimes$ denotes the dyadic or outer product of two vectors, which results in a matrix and is defined in (A.572).

We note that the Green's function (D.20) is symmetric in the sense that

$$G(\boldsymbol{x}, \boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x}), \tag{D.24}$$

and it fulfills similarly

$$\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = \nabla_{\boldsymbol{y}} G(\boldsymbol{y}, \boldsymbol{x}) = -\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) = -\nabla_{\boldsymbol{x}} G(\boldsymbol{y}, \boldsymbol{x}), \tag{D.25}$$

and

$$\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = \nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) = \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{y}, \boldsymbol{x}) = \nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} G(\boldsymbol{y}, \boldsymbol{x}). \tag{D.26}$$

D.4 Far field of the Green's function

The far field of the Green's function describes its asymptotic behavior at infinity, i.e., when $|\boldsymbol{x}| \to \infty$ and assuming that $\boldsymbol{y}$ is fixed. For this purpose, we search the terms of highest order at infinity by expanding with respect to the variable $\boldsymbol{x}$ the expressions

$$|\boldsymbol{x}-\boldsymbol{y}|^2 = |\boldsymbol{x}|^2\left(1-\frac{2\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2}+\frac{|\boldsymbol{y}|^2}{|\boldsymbol{x}|^2}\right), \tag{D.27}$$

$$|\boldsymbol{x}-\boldsymbol{y}| = |\boldsymbol{x}|\left(1-\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2}+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^2}\right)\right), \tag{D.28}$$

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{1}{|\boldsymbol{x}|}\left(1+\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|^2}+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^2}\right)\right). \tag{D.29}$$

We express the point $\boldsymbol{x}$ as $\boldsymbol{x} = |\boldsymbol{x}|\hat{\boldsymbol{x}}$, being $\hat{\boldsymbol{x}}$ a unitary vector. The far field of the Green's function, as $|\boldsymbol{x}| \to \infty$, is thus given by

$$G^{ff}(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{4\pi|\boldsymbol{x}|} - \frac{\boldsymbol{y}\cdot\hat{\boldsymbol{x}}}{4\pi|\boldsymbol{x}|^2}. \tag{D.30}$$

Similarly, as $|\boldsymbol{x}| \to \infty$, we have for its gradient with respect to $\boldsymbol{y}$, that

$$\nabla_{\boldsymbol{y}} G^{ff}(\boldsymbol{x},\boldsymbol{y}) = -\frac{\hat{\boldsymbol{x}}}{4\pi|\boldsymbol{x}|^2}, \tag{D.31}$$

for its gradient with respect to $\boldsymbol{x}$, that

$$\nabla_{\boldsymbol{x}} G^{ff}(\boldsymbol{x},\boldsymbol{y}) = \frac{\hat{\boldsymbol{x}}}{4\pi|\boldsymbol{x}|^2}, \tag{D.32}$$

and for its double-gradient matrix, that

$$\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}} G^{ff}(\boldsymbol{x},\boldsymbol{y}) = -\frac{\boldsymbol{I}}{4\pi|\boldsymbol{x}|^3} + \frac{3(\hat{\boldsymbol{x}}\otimes\hat{\boldsymbol{x}})}{4\pi|\boldsymbol{x}|^3}. \tag{D.33}$$

D.5 Transmission problem

We are interested in expressing the solution u of the direct perturbation problem (D.8) by means of an integral representation formula over the boundary Γ. To study this kind of representations, the differential problem defined on Ω_e is extended as a transmission problem defined now on the whole space $\mathbb{R}^3$ by combining (D.8) with a corresponding interior problem defined on Ω_i. For the transmission problem, which specifies jump conditions over the boundary Γ, a general integral representation can be developed, and the particular integral representations of interest are then established by the specific choice of the corresponding interior problem.

A transmission problem is then a differential problem for which the jump conditions of the solution field, rather than boundary conditions, are specified on the boundary Γ. As shown in Figure D.1, we consider the exterior domain Ω_e and the interior domain Ω_i, taking the unit normal $\boldsymbol{n}$ pointing towards Ω_i. We search now a solution u defined in $\Omega_e \cup \Omega_i$, and use the notation $u_e = u|_{\Omega_e}$ and $u_i = u|_{\Omega_i}$. We define the jumps of the traces of u on both sides of the boundary Γ as

$$[u] = u_e - u_i \qquad \text{and} \qquad \left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n}. \tag{D.34}$$

The transmission problem is now given by

$$\begin{cases} \text{Find } u : \Omega_e \cup \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u = 0 \qquad \text{in } \ \Omega_e \cup \Omega_i, \\ [u] = \mu \qquad \text{on } \ \Gamma, \\ \left[\dfrac{\partial u}{\partial n}\right] = \nu \qquad \text{on } \ \Gamma, \\ + \text{ Decaying condition as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{D.35}$$

where $\mu, \nu : \Gamma \to \mathbb{C}$ are known functions. The decaying condition is still (D.5), and it is required to ensure uniqueness of the solution.

D.6 Integral representations and equations

D.6.1 Integral representation

To develop for the solution u an integral representation formula over the boundary Γ, we define by $\Omega_{R,\varepsilon}$ the domain $\Omega_e \cup \Omega_i$ without the ball B_ε of radius $\varepsilon > 0$ centered at the point $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, and truncated at infinity by the ball B_R of radius $R > 0$ centered at the origin. We consider that the ball B_ε is entirely contained either in Ω_e or in Ω_i, depending on the location of its center $\boldsymbol{x}$. Therefore, as shown in Figure D.2, we have that

$$\Omega_{R,\varepsilon} = \big((\Omega_e \cup \Omega_i) \cap B_R\big) \setminus \overline{B_\varepsilon}, \tag{D.36}$$

where

$$B_R = \{\boldsymbol{y} \in \mathbb{R}^3 : \ |\boldsymbol{y}| < R\} \qquad \text{and} \qquad B_\varepsilon = \{\boldsymbol{y} \in \mathbb{R}^3 : \ |\boldsymbol{y} - \boldsymbol{x}| < \varepsilon\}. \tag{D.37}$$

We consider similarly the boundaries of the balls

$$S_R = \{\boldsymbol{y} \in \mathbb{R}^3 : \ |\boldsymbol{y}| = R\} \qquad \text{and} \qquad S_\varepsilon = \{\boldsymbol{y} \in \mathbb{R}^3 : \ |\boldsymbol{y} - \boldsymbol{x}| = \varepsilon\}. \tag{D.38}$$

The idea is to retrieve the domain $\Omega_e \cup \Omega_i$ at the end when the limits $R \to \infty$ and $\varepsilon \to 0$ are taken for the truncated domain $\Omega_{R,\varepsilon}$.

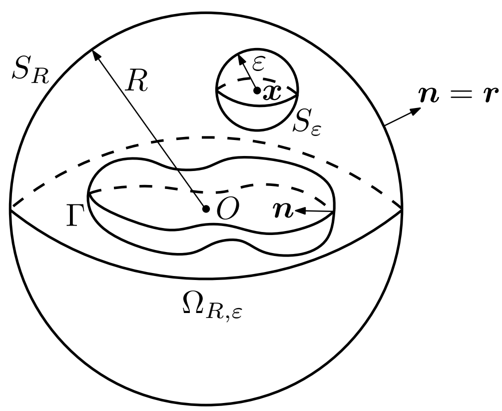

FIGURE D.2. Truncated domain $\Omega_{R,\varepsilon}$ for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$.

We apply now Green's second integral theorem (A.613) to the functions u and $G(\boldsymbol{x},\cdot)$ in the bounded domain $\Omega_{R,\varepsilon}$, yielding

$$\begin{aligned}0 &= \int_{\Omega_{R,\varepsilon}} \big(u(\boldsymbol{y})\Delta_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\Delta u(\boldsymbol{y})\big)\mathrm{d}\boldsymbol{y}\\ &= \int_{S_R} \left(u(\boldsymbol{y})\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial r}(\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y})\\ &\quad - \int_{S_\varepsilon} \left(u(\boldsymbol{y})\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial r}(\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y})\\ &\quad + \int_{\Gamma} \left([u](\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y}).\end{aligned} \tag{D.39}$$

For R large enough, the integral on S_R tends to zero, since

$$\left|\int_{S_R} u(\boldsymbol{y})\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})\right| \leq \frac{C}{R}, \tag{D.40}$$

and

$$\left|\int_{S_R} G(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial r}(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})\right| \leq \frac{C}{R}, \tag{D.41}$$

for some constants $C > 0$, due the asymptotic decaying behavior at infinity (D.5). If the function u is regular enough in the ball B_ε, then the second term of the integral on S_ε, when $\varepsilon \to 0$ and due (D.20), is bounded by

$$\left|\int_{S_\varepsilon} G(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial r}(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})\right| \leq \varepsilon \sup_{\boldsymbol{y}\in B_\varepsilon}\left|\frac{\partial u}{\partial r}(\boldsymbol{y})\right|, \tag{D.42}$$

and tends to zero. The regularity of u can be specified afterwards once the integral representation has been determined and generalized by means of density arguments. The first integral term on S_ε can be decomposed as

$$\begin{aligned}\int_{S_\varepsilon} u(\boldsymbol{y})\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) &= u(\boldsymbol{x})\int_{S_\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})\\ &\quad + \int_{S_\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{y})-u(\boldsymbol{x})\big)\,\mathrm{d}\gamma(\boldsymbol{y}),\end{aligned} \tag{D.43}$$

For the first term in the right-hand side of (D.43), by replacing (D.21), we have that

$$\int_{S_\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = 1, \tag{D.44}$$

while the second term is bounded by

$$\left|\int_{S_\varepsilon}\big(u(\boldsymbol{y})-u(\boldsymbol{x})\big)\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})\right| \leq \sup_{\boldsymbol{y}\in B_\varepsilon}|u(\boldsymbol{y})-u(\boldsymbol{x})|, \tag{D.45}$$

which tends towards zero when $\varepsilon \to 0$.

In conclusion, when the limits $R \to \infty$ and $\varepsilon \to 0$ are taken in (D.39), then the following integral representation formula holds for the solution u of the transmission problem:

$$u(\boldsymbol{x}) = \int_{\Gamma}\left([u](\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y}), \qquad \boldsymbol{x}\in\Omega_e\cup\Omega_i. \tag{D.46}$$

We observe thus that if the values of the jump of u and of its normal derivative are known on Γ, then the transmission problem (D.35) is readily solved and its solution given explicitly by (D.46), which, in terms of μ and ν, becomes

$$u(\boldsymbol{x}) = \int_\Gamma \left(\mu(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \Omega_e \cup \Omega_i. \tag{D.47}$$

To determine the values of the jumps, an adequate integral equation has to be developed, i.e., an equation whose unknowns are the traces of the solution on Γ.

An alternative way to demonstrate the integral representation (D.46) is to proceed in the sense of distributions, in the same way as done in Section B.6. Again we obtain the single layer potential

$$\left\{ G * \left[\frac{\partial u}{\partial n} \right] \delta_\Gamma \right\}(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x}, \boldsymbol{y}) \left[\frac{\partial u}{\partial n} \right](\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) \tag{D.48}$$

associated with the distribution of sources $[\partial u/\partial n]\delta_\Gamma$, and the double layer potential

$$\left\{ G * \frac{\partial}{\partial n}\big([u]\delta_\Gamma\big) \right\}(\boldsymbol{x}) = -\int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) [u](\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) \tag{D.49}$$

associated with the distribution of dipoles $\frac{\partial}{\partial n}([u]\delta_\Gamma)$. Combining properly (D.48) and (D.49) yields the desired integral representation (D.46).

We note that to obtain the gradient of the integral representation (D.46) we can pass directly the derivatives inside the integral, since there are no singularities if $\boldsymbol{x} \in \Omega_e \cup \Omega_i$. Therefore we have that

$$\nabla u(\boldsymbol{x}) = \int_\Gamma \left([u](\boldsymbol{y}) \nabla_{\boldsymbol{x}} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \left[\frac{\partial u}{\partial n} \right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}). \tag{D.50}$$

We remark also that Green's first integral theorem (A.612) implies for the solution u_i of the interior problem that

$$\int_\Gamma \frac{\partial u_i}{\partial n} \, \mathrm{d}\gamma = -\int_{\Omega_i} \Delta u_i \, \mathrm{d}\boldsymbol{x} = 0. \tag{D.51}$$

Nonetheless a three-dimensional equivalent of (B.58) does no longer apply, since this integral converges to a constant as $R \to \infty$, which is not necessarily zero.

D.6.2 Integral equation

To determine the values of the traces that conform the jumps for the transmission problem (D.35), an integral equation has to be developed. For this purpose we place the source point $\boldsymbol{x}$ on the boundary Γ and apply the same procedure as before for the integral representation (D.46), treating differently in (D.39) only the integrals on S_ε. The integrals on S_R still behave well and tend towards zero as $R \to \infty$. The Ball B_ε, though, is split in half into the two pieces $\Omega_e \cap B_\varepsilon$ and $\Omega_i \cap B_\varepsilon$, which are asymptotically separated by the tangent of the boundary if Γ is regular. Thus the associated integrals on S_ε give rise to a term $-(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x}))/2$ instead of just $-u(\boldsymbol{x})$ as before. We must notice that in this case, the integrands associated with the boundary Γ admit an integrable singularity at the point $\boldsymbol{x}$. The desired integral equation related with (D.46) is then given by

$$\frac{u_e(\boldsymbol{x})+u_i(\boldsymbol{x})}{2} = \int_\Gamma \left([u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y}) \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma. \quad \text{(D.52)}$$

By choosing adequately the boundary condition of the interior problem, and by considering also the boundary condition of the exterior problem and the jump definitions (D.34), this integral equation can be expressed in terms of only one unknown function on Γ. Thus, solving the problem (D.8) is equivalent to solve (D.52) and then replace the obtained solution in (D.46).

The integral equation holds only when the boundary Γ is regular (e.g., of class C^2). Otherwise, taking the limit $\varepsilon \to 0$ can no longer be well-defined and the result is false in general. In particular, if the boundary Γ has an angular point at $\boldsymbol{x} \in \Gamma$, then the left-hand side of the integral equation (D.52) is modified on that point according to the portion of the ball B_ε that remains inside Ω_e, analogously as was done for the two-dimensional case in (B.61), but now for solid angles.

Another integral equation can be also derived for the normal derivative of the solution u on the boundary Γ, by studying the jump properties of the single and double layer potentials. Its derivation is more complicated than for (D.52), being the specific details explicited in the subsection of boundary layer potentials. If the boundary is regular at $\boldsymbol{x} \in \Gamma$, then we obtain

$$\frac{1}{2}\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{1}{2}\frac{\partial u_i}{\partial n}(\boldsymbol{x}) = \int_\Gamma \left([u](\boldsymbol{y}) \frac{\partial^2 G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}). \quad \text{(D.53)}$$

This integral equation is modified correspondingly if $\boldsymbol{x}$ is an angular point.

D.6.3 Integral kernels

In the same manner as in the two-dimensional case, the integral kernels G, $\partial G/\partial n_{\boldsymbol{y}}$, and $\partial G/\partial n_{\boldsymbol{x}}$ are weakly singular, and thus integrable, whereas the kernel $\partial^2 G/\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}$ is not integrable and therefore hypersingular.

The kernel G defined in (D.20) fulfills evidently (B.64) with $\lambda = 1$. On the other hand, the kernels $\partial G/\partial n_{\boldsymbol{y}}$ and $\partial G/\partial n_{\boldsymbol{x}}$ are less singular along Γ than they appear at first sight, due the regularizing effect of the normal derivatives. They are given respectively by

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \frac{(\boldsymbol{y}-\boldsymbol{x})\cdot \boldsymbol{n}_{\boldsymbol{y}}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|^3} \quad \text{and} \quad \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) = \frac{(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{n}_{\boldsymbol{x}}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}. \quad \text{(D.54)}$$

It can be shown that the estimates (B.70) and (B.71) hold also in three dimensions, by using the same reasoning as in the two-dimensional case for the graph of a regular function φ that takes variables now on the tangent plane. Therefore we have that

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{O}\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right) \quad \text{and} \quad \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{O}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right), \quad \text{(D.55)}$$

and hence these kernels satisfy (B.64) with $\lambda = 1$.

The kernel $\partial^2 G/\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}$, on the other hand, adopts the form

$$\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = -\frac{\boldsymbol{n}_{\boldsymbol{x}}\cdot\boldsymbol{n}_{\boldsymbol{y}}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|^3} - \frac{3\big((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}_{\boldsymbol{x}}\big)\big((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}\big)}{4\pi|\boldsymbol{y}-\boldsymbol{x}|^5}. \tag{D.56}$$

The regularizing effect of the normal derivatives applies only to its second term, but not to the first. Hence this kernel is hypersingular, with $\lambda = 3$, and it holds that

$$\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{O}\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|^3}\right). \tag{D.57}$$

The kernel is no longer integrable and the associated integral operator has to be thus interpreted in some appropriate sense as a divergent integral (cf., e.g., Hsiao & Wendland 2008, Lenoir 2005, Nédélec 2001).

D.6.4 Boundary layer potentials

We regard now the jump properties on the boundary Γ of the boundary layer potentials that have appeared in our calculations. For the development of the integral representation (D.47) we already made acquaintance with the single and double layer potentials, which we define now more precisely for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$ as the integral operators

$$\mathcal{S}\nu(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{D.58}$$

$$\mathcal{D}\mu(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.59}$$

The integral representation (D.47) can be now stated in terms of the layer potentials as

$$u = \mathcal{D}\mu - \mathcal{S}\nu. \tag{D.60}$$

We remark that for any functions $\nu, \mu : \Gamma \to \mathbb{C}$ that are regular enough, the single and double layer potentials satisfy the Laplace equation, namely

$$\Delta\mathcal{S}\nu = 0 \qquad \text{in } \ \Omega_e \cup \Omega_i, \tag{D.61}$$

$$\Delta\mathcal{D}\mu = 0 \qquad \text{in } \ \Omega_e \cup \Omega_i. \tag{D.62}$$

For the integral equations (D.52) and (D.53), which are defined for $\boldsymbol{x} \in \Gamma$, we require the four boundary integral operators:

$$S\nu(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{D.63}$$

$$D\mu(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{D.64}$$

$$D^*\nu(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{D.65}$$

$$N\mu(\boldsymbol{x}) = \int_\Gamma \frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.66}$$

The operator D^* is in fact the adjoint of the operator D. As we already mentioned, the kernel of the integral operator N defined in (D.66) is not integrable, yet we write it formally as an improper integral. An appropriate sense for this integral will be given below. The integral equations (D.52) and (D.53) can be now stated in terms of the integral operators as

$$\frac{1}{2}(u_e + u_i) = D\mu - S\nu, \tag{D.67}$$

$$\frac{1}{2}\left(\frac{\partial u_e}{\partial n} + \frac{\partial u_i}{\partial n}\right) = N\mu - D^*\nu. \tag{D.68}$$

These integral equations can be easily derived from the jump properties of the single and double layer potentials. The single layer potential (D.58) is continuous and its normal derivative has a jump of size $-\nu$ across Γ, i.e.,

$$\mathcal{S}\nu|_{\Omega_e} = S\nu = \mathcal{S}\nu|_{\Omega_i}, \tag{D.69}$$

$$\frac{\partial}{\partial n}\mathcal{S}\nu|_{\Omega_e} = \left(-\frac{1}{2} + D^*\right)\nu, \tag{D.70}$$

$$\frac{\partial}{\partial n}\mathcal{S}\nu|_{\Omega_i} = \left(\frac{1}{2} + D^*\right)\nu. \tag{D.71}$$

The double layer potential (D.59), on the other hand, has a jump of size μ across Γ and its normal derivative is continuous, namely

$$\mathcal{D}\mu|_{\Omega_e} = \left(\frac{1}{2} + D\right)\mu, \tag{D.72}$$

$$\mathcal{D}\mu|_{\Omega_i} = \left(-\frac{1}{2} + D\right)\mu, \tag{D.73}$$

$$\frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_e} = N\mu = \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_i}. \tag{D.74}$$

The integral equation (D.67) is obtained directly either from (D.69) and (D.72), or from (D.69) and (D.73), by considering the appropriate trace of (D.60) and by defining the functions μ and ν as in (D.35). These three jump properties are easily proven by regarding the details of the proof for (D.52).

Similarly, the integral equation (D.68) for the normal derivative is obtained directly either from (D.70) and (D.74), or from (D.71) and (D.74), by considering the appropriate trace of the normal derivative of (D.60) and by defining again the functions μ and ν as in (D.35). The proof of these other three jump properties is done below.

a) Jump of the normal derivative of the single layer potential

Let us then study first the proof of (D.70) and (D.71). The traces of the normal derivative of the single layer potential are given by

$$\frac{\partial}{\partial n}\mathcal{S}\nu(\boldsymbol{x})|_{\Omega_e} = \lim_{\Omega_e \ni \boldsymbol{z} \to \boldsymbol{x}} \nabla\mathcal{S}\nu(\boldsymbol{z}) \cdot \boldsymbol{n_x}, \tag{D.75}$$

$$\frac{\partial}{\partial n}\mathcal{S}\nu(\boldsymbol{x})|_{\Omega_i} = \lim_{\Omega_i \ni \boldsymbol{z} \to \boldsymbol{x}} \nabla\mathcal{S}\nu(\boldsymbol{z}) \cdot \boldsymbol{n_x}. \tag{D.76}$$

Now we have that

$$\nabla \mathcal{S}\nu(\boldsymbol{z}) \cdot \boldsymbol{n_x} = \int_\Gamma \boldsymbol{n_x} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.77}$$

For $\varepsilon > 0$ we denote $\Gamma_\varepsilon = \Gamma \cap B_\varepsilon$, i.e., the portion of Γ contained inside the ball B_ε of radius ε and centered at $\boldsymbol{x}$. By decomposing the integral we obtain that

$$\nabla \mathcal{S}\nu(\boldsymbol{z}) \cdot \boldsymbol{n_x} = \int_{\Gamma\setminus\Gamma_\varepsilon} \boldsymbol{n_x} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) + \int_{\Gamma_\varepsilon} \boldsymbol{n_x} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.78}$$

For the first integral in (D.78) we can take without problems the limit $\boldsymbol{z} \to \boldsymbol{x}$, since for a fixed ε the integral is regular in $\boldsymbol{x}$. Since the singularity of the resulting kernel $\partial G/\partial n_{\boldsymbol{x}}$ is integrable, Lebesgue's dominated convergence theorem (cf. Royden 1988) implies that

$$\lim_{\varepsilon\to 0} \int_{\Gamma\setminus\Gamma_\varepsilon} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = D^*\nu(\boldsymbol{x}). \tag{D.79}$$

Let us treat now the second integral in (D.78), which is again decomposed in different integrals in such a way that

$$\begin{aligned}\int_{\Gamma_\varepsilon} \boldsymbol{n_x} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) &= \int_{\Gamma_\varepsilon} (\boldsymbol{n_x} - \boldsymbol{n_y}) \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \\ &+ \int_{\Gamma_\varepsilon} \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\big(\nu(\boldsymbol{y}) - \nu(\boldsymbol{x})\big)\,\mathrm{d}\gamma(\boldsymbol{y}) + \nu(\boldsymbol{x}) \int_{\Gamma_\varepsilon} \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \end{aligned} \tag{D.80}$$

When ε is small, and since Γ is supposed to be regular, therefore Γ_ε resembles a flat disc of radius ε. Thus we have that

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} (\boldsymbol{n_x} - \boldsymbol{n_y}) \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = 0. \tag{D.81}$$

If ν is regular enough, then we have also that

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\big(\nu(\boldsymbol{y}) - \nu(\boldsymbol{x})\big)\,\mathrm{d}\gamma(\boldsymbol{y}) = 0. \tag{D.82}$$

For the remaining term in (D.80) we consider the solid angle Θ under which the almost flat disc Γ_ε is seen from point $\boldsymbol{z}$ (cf. Figure D.3). If we denote $\boldsymbol{R} = \boldsymbol{y} - \boldsymbol{z}$ and $R = |\boldsymbol{R}|$, and consider an oriented surface differential element $\mathrm{d}\boldsymbol{\gamma} = \boldsymbol{n_y}\mathrm{d}\gamma(\boldsymbol{y})$ seen from point $\boldsymbol{z}$, then we can express the solid angle differential element by (cf. Terrasse & Abboud 2006)

$$\mathrm{d}\Theta = \frac{\boldsymbol{R}}{R^3} \cdot \mathrm{d}\boldsymbol{\gamma} = \frac{\boldsymbol{R}\cdot\boldsymbol{n_y}}{R^3}\,\mathrm{d}\gamma(\boldsymbol{y}) = 4\pi\,\boldsymbol{n_y} \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.83}$$

Integrating over the disc Γ_ε and considering (D.25) yields the solid angle Θ, namely

$$\Theta = \int_{\Gamma_\varepsilon} \mathrm{d}\Theta = 4\pi \int_{\Gamma_\varepsilon} \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = -4\pi \int_{\Gamma_\varepsilon} \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{D.84}$$

where $-2\pi \le \Theta \le 2\pi$. The solid angle Θ is positive when the vectors $\boldsymbol{R}$ and $\boldsymbol{n_y}$ point towards the same side of Γ_ε, and negative when they oppose each other. Thus if $\boldsymbol{z}$ is very close to $\boldsymbol{x}$ and if ε is small enough so that Γ_ε behaves as a flat disc, then

$$\int_{\Gamma_\varepsilon} \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \approx \begin{cases} -1/2 & \text{if } \boldsymbol{z} \in \Omega_e, \\ 1/2 & \text{if } \boldsymbol{z} \in \Omega_i. \end{cases} \tag{D.85}$$

Hence we obtain the desired jump formulae (D.70) and (D.71).

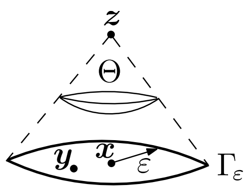

FIGURE D.3. Solid angle under which Γ_ε is seen from point $\boldsymbol{z}$.

b) Continuity of the normal derivative of the double layer potential

We are now interested in proving the continuity of the normal derivative of the double layer potential across Γ, as expressed in (D.74). This will allow us at the same time to define an appropriate sense for the improper integral (D.66). This integral is divergent in a classical sense, but it can be nonetheless properly defined in a weak or distributional sense by considering it as a linear functional acting on a test function $\varphi \in \mathcal{D}(\mathbb{R}^3)$. By considering (D.62) and Green's first integral theorem (A.612), we can express our values of interest in a weak sense as

$$\left\langle \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_e}, \varphi \right\rangle = \int_\Gamma \frac{\partial}{\partial n}\mathcal{D}\mu(\boldsymbol{x})|_{\Omega_e}\, \varphi(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) = \int_{\Omega_e} \nabla\mathcal{D}\mu(\boldsymbol{x})\cdot\nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{D.86}$$

$$\left\langle \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_i}, \varphi \right\rangle = \int_\Gamma \frac{\partial}{\partial n}\mathcal{D}\mu(\boldsymbol{x})|_{\Omega_i}\, \varphi(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}) = -\int_{\Omega_i} \nabla\mathcal{D}\mu(\boldsymbol{x})\cdot\nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{D.87}$$

From (A.588) and (D.25) we obtain the relation

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{n_y}\cdot\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{n_y}\cdot\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y}) = -\operatorname{div}_{\boldsymbol{x}}\big(G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n_y}\big). \tag{D.88}$$

Thus for the double layer potential (D.59) we have that

$$\mathcal{D}\mu(\boldsymbol{x}) = -\operatorname{div}\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n_y}\,\mathrm{d}\gamma(\boldsymbol{y}) = -\operatorname{div}\mathcal{S}(\mu\,\boldsymbol{n_y})(\boldsymbol{x}), \tag{D.89}$$

being its gradient given by

$$\nabla\mathcal{D}\mu(\boldsymbol{x}) = -\nabla\operatorname{div}\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n_y}\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.90}$$

From (A.589) we have that

$$\operatorname{curl}_{\boldsymbol{x}}\big(G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n_y}\big) = \nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\times\boldsymbol{n_y}. \tag{D.91}$$

Hence, by considering (A.590), (D.62), and (D.91) in (D.90), we obtain that

$$\nabla\mathcal{D}\mu(\boldsymbol{x}) = \operatorname{curl}\int_\Gamma \big(\boldsymbol{n_y}\times\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\big)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.92}$$

From (D.25) and (A.658) we have that

$$\int_\Gamma \big(\boldsymbol{n_y}\times\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\big)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = -\int_\Gamma \boldsymbol{n_y}\times\big(\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\big)\,\mathrm{d}\gamma(\boldsymbol{y})$$
$$= \int_\Gamma \boldsymbol{n_y}\times\big(G(\boldsymbol{x},\boldsymbol{y})\nabla\mu(\boldsymbol{y})\big)\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{D.93}$$

and consequently

$$\nabla\mathcal{D}\mu(\boldsymbol{x}) = \operatorname{curl}\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{n_y}\times\nabla\mu(\boldsymbol{y})\big)\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.94}$$

Now, considering (A.596) and (A.618), and replacing (D.94) in (D.86), implies that

$$\int_{\Omega_e}\nabla\mathcal{D}\mu(\boldsymbol{x})\cdot\nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y})\times\boldsymbol{n_y}\big)\cdot\big(\nabla\varphi(\boldsymbol{x})\times\boldsymbol{n_x}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}). \tag{D.95}$$

Analogously, when replacing in (D.87) we have that

$$\int_{\Omega_i}\nabla\mathcal{D}\mu(\boldsymbol{x})\cdot\nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y})\times\boldsymbol{n_y}\big)\cdot\big(\nabla\varphi(\boldsymbol{x})\times\boldsymbol{n_x}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}). \tag{D.96}$$

Hence, from (D.86), (D.87), (D.95), and (D.96) we conclude the proof of (D.74). The integral operator (D.66) is thus properly defined in a weak sense for $\varphi\in\mathcal{D}(\mathbb{R}^3)$ by

$$\langle N\mu(\boldsymbol{x}),\,\varphi\rangle = -\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y})\times\boldsymbol{n_y}\big)\cdot\big(\nabla\varphi(\boldsymbol{x})\times\boldsymbol{n_x}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}). \tag{D.97}$$

D.6.5 Alternatives for integral representations and equations

By taking into account the transmission problem (D.35), its integral representation formula (D.46), and its integral equations (D.52) and (D.53), several particular alternatives for integral representations and equations of the exterior problem (D.8) can be developed. The way to perform this is to extend properly the exterior problem towards the interior domain Ω_i, either by specifying explicitly this extension or by defining an associated interior problem, so as to become the desired jump properties across Γ. The extension has to satisfy the Laplace equation (D.1) in Ω_i and a boundary condition that corresponds adequately to the impedance boundary condition (D.2). The obtained system of integral representations and equations allows finally to solve the exterior problem (D.8), by using the solution of the integral equation in the integral representation formula.

a) Extension by zero

An extension by zero towards the interior domain Ω_i implies that

$$u_i = 0 \qquad \text{in } \Omega_i. \tag{D.98}$$

The jumps over Γ are characterized in this case by

$$[u] = u_e = \mu, \tag{D.99}$$
$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} = Zu_e - f_z = Z\mu - f_z, \tag{D.100}$$

where $\mu : \Gamma \to \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by

$$u(\boldsymbol{x}) = \int_\Gamma \left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y}) \right) \mu(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y}) + \int_\Gamma G(\boldsymbol{x},\boldsymbol{y}) f_z(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y}). \quad \text{(D.101)}$$

Since

$$\frac{1}{2}\big(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x})\big) = \frac{\mu(\boldsymbol{x})}{2}, \qquad \boldsymbol{x} \in \Gamma, \quad \text{(D.102)}$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\frac{\mu(\boldsymbol{x})}{2} + \int_\Gamma \left(Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y}) - \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) \right) \mu(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y}) f_z(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y}), \quad \text{(D.103)}$$

which has to be solved for the unknown μ. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) - \mathcal{S}(Z\mu) + \mathcal{S}(f_z) \qquad \text{in } \Omega_e \cup \Omega_i, \quad \text{(D.104)}$$

$$\frac{\mu}{2} + S(Z\mu) - D(\mu) = S(f_z) \qquad \text{on } \Gamma. \quad \text{(D.105)}$$

Alternatively, since

$$\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{\partial u_i}{\partial n}(\boldsymbol{x}) \right) = \frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x}) - \frac{f_z(\boldsymbol{x})}{2}, \qquad \boldsymbol{x} \in \Gamma, \quad \text{(D.106)}$$

we obtain also, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x}) + \int_\Gamma \left(-\frac{\partial^2 G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) + Z(\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \right) \mu(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y})$$
$$= \frac{f_z(\boldsymbol{x})}{2} + \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) f_z(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y}), \quad \text{(D.107)}$$

which in terms of boundary layer potentials becomes

$$\frac{Z}{2}\mu - N(\mu) + D^*(Z\mu) = \frac{f_z}{2} + D^*(f_z) \qquad \text{on } \Gamma. \quad \text{(D.108)}$$

b) Continuous impedance

We associate to (D.8) the interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i = 0 \qquad \text{in } \Omega_i, \\ -\dfrac{\partial u_i}{\partial n} + Z u_i = f_z \qquad \text{on } \Gamma. \end{cases} \quad \text{(D.109)}$$

The jumps over Γ are characterized in this case by

$$[u] = u_e - u_i = \mu, \quad \text{(D.110)}$$

$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = Z(u_e - u_i) = Z\mu, \quad \text{(D.111)}$$

where $\mu : \Gamma \to \mathbb{C}$ is a function to be determined. In particular it holds that the jump of the impedance is zero, namely

$$\left[-\frac{\partial u}{\partial n} + Zu\right] = \left(-\frac{\partial u_e}{\partial n} + Zu_e\right) - \left(-\frac{\partial u_i}{\partial n} + Zu_i\right) = 0. \tag{D.112}$$

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by

$$u(\boldsymbol{x}) = \int_\Gamma \left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.113}$$

Since

$$-\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{\partial u_i}{\partial n}(\boldsymbol{x})\right) + \frac{Z(\boldsymbol{x})}{2}\big(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x})\big) = f_z(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma, \tag{D.114}$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the first kind

$$\int_\Gamma \left(-\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) + Z(\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \\ + Z(\boldsymbol{x})\int_\Gamma \left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = f_z(\boldsymbol{x}), \tag{D.115}$$

which has to be solved for the unknown μ. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) - \mathcal{S}(Z\mu) \qquad \text{in } \Omega_e \cup \Omega_i, \tag{D.116}$$

$$-N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu) = f_z \qquad \text{on } \Gamma. \tag{D.117}$$

We observe that the integral equation (D.117) is self-adjoint.

c) Continuous value

We associate to (D.8) the interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i = 0 & \text{in } \Omega_i, \\ -\dfrac{\partial u_e}{\partial n} + Zu_i = f_z & \text{on } \Gamma. \end{cases} \tag{D.118}$$

The jumps over Γ are characterized in this case by

$$[u] = u_e - u_i = \frac{1}{Z}\left(\frac{\partial u_e}{\partial n} - f_z\right) - \frac{1}{Z}\left(\frac{\partial u_e}{\partial n} - f_z\right) = 0, \tag{D.119}$$

$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = \nu, \tag{D.120}$$

where $\nu : \Gamma \to \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by the single layer potential

$$u(\boldsymbol{x}) = -\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{D.121}$$

Since

$$-\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x})+\frac{\partial u_i}{\partial n}(\boldsymbol{x})\right)+\frac{Z(\boldsymbol{x})}{2}\big(u_e(\boldsymbol{x})+u_i(\boldsymbol{x})\big)=\frac{\nu(\boldsymbol{x})}{2}+f_z(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma,\tag{D.122}$$

we obtain, for $\boldsymbol{x}\in\Gamma$, the Fredholm integral equation of the second kind

$$-\frac{\nu(\boldsymbol{x})}{2}+\int_\Gamma\left(\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})-Z(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{y})\right)\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})=f_z(\boldsymbol{x}),\tag{D.123}$$

which has to be solved for the unknown ν. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u=-\mathcal{S}(\nu)\qquad\text{in }\ \Omega_e\cup\Omega_i,\tag{D.124}$$

$$\frac{\nu}{2}+ZS(\nu)-D^*(\nu)=-f_z\qquad\text{on }\ \Gamma.\tag{D.125}$$

We observe that the integral equation (D.125) is mutually adjoint with (D.105).

d) Continuous normal derivative

We associate to (D.8) the interior problem

$$\begin{cases}\text{Find } u_i:\Omega_i\to\mathbb{C}\text{ such that}\\ \Delta u_i=0 & \text{in }\ \Omega_i,\\ -\dfrac{\partial u_i}{\partial n}+Zu_e=f_z & \text{on }\ \Gamma.\end{cases}\tag{D.126}$$

The jumps over Γ are characterized in this case by

$$[u]=u_e-u_i=\mu,\tag{D.127}$$

$$\left[\frac{\partial u}{\partial n}\right]=\frac{\partial u_e}{\partial n}-\frac{\partial u_i}{\partial n}=\big(Zu_e-f_z\big)-\big(Zu_e-f_z\big)=0,\tag{D.128}$$

where $\mu:\Gamma\to\mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\boldsymbol{x}\in\Omega_e\cup\Omega_i$, is given by the double layer potential

$$u(\boldsymbol{x})=\int_\Gamma\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}).\tag{D.129}$$

Since when $\boldsymbol{x}\in\Gamma$,

$$-\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x})+\frac{\partial u_i}{\partial n}(\boldsymbol{x})\right)+\frac{Z(\boldsymbol{x})}{2}\big(u_e(\boldsymbol{x})+u_i(\boldsymbol{x})\big)=-\frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x})+f_z(\boldsymbol{x}),\tag{D.130}$$

we obtain, for $\boldsymbol{x}\in\Gamma$, the Fredholm integral equation of the second kind

$$\frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x})+\int_\Gamma\left(-\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})+Z(\boldsymbol{x})\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})=f_z(\boldsymbol{x}),\tag{D.131}$$

which has to be solved for the unknown μ. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u=\mathcal{D}(\mu)\qquad\text{in }\ \Omega_e\cup\Omega_i,\tag{D.132}$$

$$\frac{Z}{2}\mu - N(\mu) + ZD(\mu) = f_z \qquad \text{on } \Gamma. \tag{D.133}$$

We observe that the integral equation (D.133) is mutually adjoint with (D.108).

D.7 Far field of the solution

The asymptotic behavior at infinity of the solution u of (D.8) is described by the far field u^{ff}. Its expression can be deduced by replacing the far field of the Green's function G^{ff} and its derivatives in the integral representation formula (D.46), which yields

$$u^{ff}(\boldsymbol{x}) = \int_\Gamma \left([u](\boldsymbol{y}) \frac{\partial G^{ff}}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G^{ff}(\boldsymbol{x},\boldsymbol{y}) \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}). \tag{D.134}$$

By replacing now (D.30) and (D.31) in (D.134), we have that the far field of the solution is

$$u^{ff}(\boldsymbol{x}) = -\frac{1}{4\pi|\boldsymbol{x}|^2} \int_\Gamma \left(\hat{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}\, [u](\boldsymbol{y}) - \hat{\boldsymbol{x}} \cdot \boldsymbol{y} \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) + \frac{1}{4\pi|\boldsymbol{x}|} \int_\Gamma \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\, \mathrm{d}\gamma(\boldsymbol{y}). \tag{D.135}$$

The asymptotic behavior of the solution u at infinity is therefore given by

$$u(\boldsymbol{x}) = \frac{C}{|\boldsymbol{x}|} + \frac{u_\infty(\hat{\boldsymbol{x}})}{|\boldsymbol{x}|^2} + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^3}\right), \qquad |\boldsymbol{x}| \to \infty, \tag{D.136}$$

uniformly in all directions $\hat{\boldsymbol{x}}$ on the unit sphere, where C is a constant, given by

$$C = \frac{1}{4\pi} \int_\Gamma \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\, \mathrm{d}\gamma(\boldsymbol{y}), \tag{D.137}$$

and where

$$u_\infty(\hat{\boldsymbol{x}}) = -\frac{1}{4\pi} \int_\Gamma \left(\hat{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}\, [u](\boldsymbol{y}) - \hat{\boldsymbol{x}} \cdot \boldsymbol{y} \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) \tag{D.138}$$

is called the far-field pattern of u. It can be expressed in decibels (dB) by means of the asymptotic cross section

$$Q_s(\hat{\boldsymbol{x}}) \ \ [\mathrm{dB}] = 20 \log_{10}\left(\frac{|u_\infty(\hat{\boldsymbol{x}})|}{|u_0|} \right), \tag{D.139}$$

where the reference level u_0 may typically depend on u_W, but for simplicity we take $u_0 = 1$.

We remark that the far-field behavior (D.136) of the solution is in accordance with the decaying condition (D.5), which justifies its choice.

D.8 Exterior sphere problem

To understand better the resolution of the direct perturbation problem (D.8), we study now the particular case when the domain $\Omega_e \subset \mathbb{R}^3$ is taken as the exterior of a sphere of radius $R > 0$. The interior of the sphere is then given by $\Omega_i = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < R\}$ and its boundary by $\Gamma = \partial\Omega_e$, as shown in Figure D.4. We place the origin at the center of Ω_i and we consider that the unit normal $\boldsymbol{n}$ is taken outwardly oriented of Ω_e, i.e., $\boldsymbol{n} = -\boldsymbol{r}$.

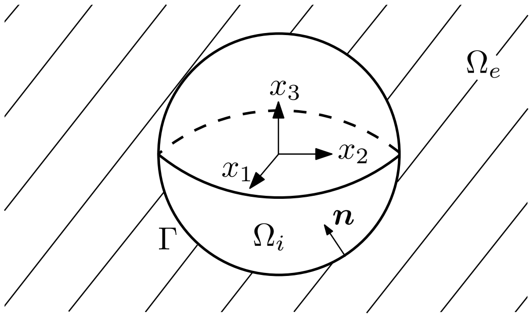

FIGURE D.4. Exterior of the sphere.

The exterior sphere problem is then stated as

$$\begin{cases} \text{Find } u : \Omega_e \to \mathbb{C} \text{ such that} \\ \Delta u = 0 & \text{in } \Omega_e, \\ \dfrac{\partial u}{\partial r} + Zu = f_z & \text{on } \Gamma, \\ \text{+ Decaying condition as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{D.140}$$

where we consider a constant impedance $Z \in \mathbb{C}$ and where the asymptotic decaying condition is as usual given by (D.5).

Due the particular chosen geometry, the solution u of (D.140) can be easily found analytically by using the method of variable separation, i.e., by supposing that

$$u(\boldsymbol{x}) = u(r, \theta, \varphi) = \frac{h(r)}{r} g(\theta) f(\varphi), \tag{D.141}$$

where the radius $r \geq 0$, the polar angle $0 \leq \theta \leq \pi$, and the azimuthal angle $-\pi < \varphi \leq \pi$ denote the spherical coordinates in $\mathbb{R}^3$, which are characterized by

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right), \qquad \varphi = \arctan\left(\frac{y}{x}\right). \tag{D.142}$$

If the Laplace equation in (D.140) is expressed using spherical coordinates, then

$$\Delta u = \frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\!\theta} \frac{\partial^2 u}{\partial \varphi^2} = 0. \tag{D.143}$$

By replacing now (D.141) in (D.143) we obtain

$$\frac{h''(r)}{r} g(\theta) f(\varphi) + \frac{h(r) f(\varphi)}{r^3 \sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin\theta \frac{\mathrm{d}g}{\mathrm{d}\theta}(\theta) \right) + \frac{h(r) g(\theta) f''(\varphi)}{r^3 \sin^2\!\theta} = 0. \tag{D.144}$$

Multiplying by $r^3 \sin^2\!\theta$, dividing by hgf, and rearranging yields

$$r^2 \sin^2\!\theta \left[\frac{h''(r)}{h(r)} + \frac{1}{g(\theta) r^2 \sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin\theta \frac{\mathrm{d}g}{\mathrm{d}\theta}(\theta) \right) \right] + \frac{f''(\varphi)}{f(\varphi)} = 0. \tag{D.145}$$

The dependence on φ has now been isolated in the last term. Consequently this term must be equal to a constant, which for convenience we denote by $-m^2$, i.e.,

$$\frac{f''(\varphi)}{f(\varphi)} = -m^2. \tag{D.146}$$

The solution of (D.146), up to a multiplicative constant, is of the form

$$f(\varphi) = e^{\pm im\varphi}. \tag{D.147}$$

For $f(\varphi)$ to be single-valued, m must be an integer if the full azimuthal range is allowed. By similar considerations we find the following separate equations for $g(\theta)$ and $h(r)$:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}g}{\mathrm{d}\theta}(\theta)\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right) g(\theta) = 0, \tag{D.148}$$

$$r^2 h''(r) - l(l+1)h(r) = 0, \tag{D.149}$$

where $l(l+1)$ is another conveniently denoted real constant. The solution $h(r)$ of the radial equation (D.149) is easily found to be

$$h(r) = a_l\, r^{l-1} + b_l\, r^{-l}, \tag{D.150}$$

where $a_l, b_l \in \mathbb{C}$ are arbitrary constants and where l is still undetermined. For the equation of the polar angle θ we consider the change of variables $x = \cos\theta$. In this case (D.148) turns into

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}g}{\mathrm{d}x}(x)\right) + \left(l(l+1) - \frac{m^2}{1-x^2}\right) g(x) = 0, \tag{D.151}$$

which corresponds to the generalized or associated Legendre differential equation (A.323), whose solutions on the interval $-1 \leq x \leq 1$ are the associated Legendre functions P_l^m and Q_l^m, which are characterized respectively by (A.330) and (A.331). If the solution is to be single-valued, finite, and continuous in $-1 \leq x \leq 1$, then we have to exclude the solutions Q_l^m, take l as a positive integer or zero, and admit for the integer m only the values $-l, -(l-1), \ldots, 0, \ldots, (l-1), l$. The solution of (D.148), up to an arbitrary multiplicative constant, is therefore given by

$$g(\theta) = P_l^m(\cos\theta). \tag{D.152}$$

It is practical to combine the angular factors $g(\theta)$ and $f(\varphi)$ into orthonormal functions over the unit sphere, the so-called spherical harmonics $Y_l^m(\theta,\varphi)$, which are defined in (A.380). The general solution for the Laplace equation considers the linear combination of all the solutions in the form (D.141), namely

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\, r^l + B_{lm}\, r^{-(l+1)}\right) Y_l^m(\theta,\varphi), \tag{D.153}$$

for some undetermined arbitrary constants $A_{lm}, B_{lm} \in \mathbb{C}$. The decaying condition (D.5) implies that

$$A_{lm} = 0, \quad -l \leq m \leq l, \quad l \geq 0. \tag{D.154}$$

Thus the general solution (D.153) turns into

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm}\, r^{-(l+1)}\, Y_l^m(\theta,\varphi), \tag{D.155}$$

and its radial derivative is given by

$$\frac{\partial u}{\partial r}(r,\theta,\varphi) = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l} (l+1) B_{lm}\, r^{-(l+2)}\, Y_l^m(\theta,\varphi). \tag{D.156}$$

The constants B_{lm} in (D.155) are determined through the impedance boundary condition on Γ. For this purpose, we expand the impedance data function f_z into spherical harmonics:

$$f_z(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}\, Y_l^m(\theta,\varphi), \qquad 0 \le \theta \le \pi, \;\; -\pi < \varphi \le \pi, \tag{D.157}$$

where

$$f_{lm} = \int_{-\pi}^{\pi}\int_0^{\pi} f_z(\theta,\varphi)\,\overline{Y_l^m(\theta,\varphi)}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi, \quad m\in\mathbb{Z}, \;\; -l \le m \le l. \tag{D.158}$$

The impedance boundary condition considers $r = R$ and thus takes the form

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{ZR-(l+1)}{R^{l+2}}\right) B_{lm}\, Y_l^m(\theta,\varphi) = f_z(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}\, Y_l^m(\theta,\varphi). \tag{D.159}$$

We observe that the constants B_{lm} can be uniquely determined only if $ZR \neq (l+1)$ for $l \in \mathbb{N}_0$. If this condition is not fulfilled, then the solution is no longer unique. Therefore, if we suppose that $ZR \neq (l+1)$ for $l \in \mathbb{N}_0$, then

$$B_{lm} = \frac{R^{l+2} f_{lm}}{ZR-(l+1)}. \tag{D.160}$$

The unique solution for the exterior sphere problem (D.140) is then given by

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{R^{l+2} f_{lm}}{ZR-(l+1)}\right) r^{-(l+1)}\, Y_l^m(\theta,\varphi). \tag{D.161}$$

We remark that there is no need here for an additional compatibility condition like (B.191).

If we consider now the case when $ZR = (n+1)$, for some particular integer $n \in \mathbb{N}_0$, then the solution u is not unique. The constants B_{nm} for $-n \le m \le n$ are then no longer defined by (D.160), and can be chosen in an arbitrary manner. For the existence of a solution in this case, however, we require also the orthogonality conditions $f_{nm} = 0$ for $-n \le m \le n$, which are equivalent to

$$\int_{-\pi}^{\pi}\int_0^{\pi} f_z(\theta,\varphi)\,\overline{Y_n^m(\theta,\varphi)}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 0, \quad -n \le m \le n. \tag{D.162}$$

Instead of (D.161), the solution of (D.140) is now given by the infinite family of functions

$$u(r,\theta,\varphi) = \sum_{0\le l\neq n}\sum_{m=-l}^{l}\left(\frac{R^{l+2} f_{lm}}{ZR-(l+1)}\right) r^{-(l+1)} Y_l^m(\theta,\varphi) + \sum_{m=-n}^{n} \frac{\alpha_m}{r^{n+1}}\, Y_n^m(\theta,\varphi), \tag{D.163}$$

where $\alpha_m \in \mathbb{C}$ for $-n \leq m \leq n$ are arbitrary and where their associated terms have the form of surface waves, i.e., waves that propagate along Γ and decrease towards the interior of Ω_e. The exterior sphere problem (D.140) admits thus a unique solution u, except on a countable set of values for ZR. And even in this last case there exists a solution, although not unique, if $2n+1$ orthogonality conditions are additionally satisfied. This behavior for the existence and uniqueness of the solution is typical of the Fredholm alternative, which applies when solving problems that involve compact perturbations of invertible operators.

D.9 Existence and uniqueness

D.9.1 Function spaces

To state a precise mathematical formulation of the herein treated problems, we have to define properly the involved function spaces. For the associated interior problems defined on the bounded set Ω_i we use the classical Sobolev space (vid. Section A.4)

$$H^1(\Omega_i) = \big\{ v : \ v \in L^2(\Omega_i), \ \nabla v \in L^2(\Omega_i)^3 \big\}, \tag{D.164}$$

which is a Hilbert space and has the norm

$$\|v\|_{H^1(\Omega_i)} = \Big(\|v\|^2_{L^2(\Omega_i)} + \|\nabla v\|^2_{L^2(\Omega_i)^3} \Big)^{1/2}. \tag{D.165}$$

For the exterior problem defined on the unbounded domain Ω_e, on the other hand, we introduce the weighted Sobolev space (cf. Nédélec 2001)

$$W^1(\Omega_e) = \left\{ v : \ \frac{v}{(1+r^2)^{1/2}} \in L^2(\Omega_e), \ \frac{\partial v}{\partial x_i} \in L^2(\Omega_e) \ \ \forall i \in \{1,2,3\} \right\}, \tag{D.166}$$

where $r = |\boldsymbol{x}|$. If $W^1(\Omega_e)$ is provided with the norm

$$\|v\|_{W^1(\Omega_e)} = \left(\left\| \frac{v}{(1+r^2)^{1/2}} \right\|^2_{L^2(\Omega_e)} + \|\nabla v\|^2_{L^2(\Omega_e)^3} \right)^{1/2}, \tag{D.167}$$

then it becomes a Hilbert space. The restriction to any bounded open set $B \subset \Omega_e$ of the functions of $W^1(\Omega_e)$ belongs to $H^1(B)$, i.e., we have the inclusion $W^1(\Omega_e) \subset H^1_{\text{loc}}(\Omega_e)$, and the functions in these two spaces differ only by their behavior at infinity. We remark that the space $W^1(\Omega_e)$ contains the constant functions and all the functions of $H^1_{\text{loc}}(\Omega_e)$ that satisfy the decaying condition (D.5).

When dealing with Sobolev spaces, even a strong Lipschitz boundary $\Gamma \in C^{0,1}$ is admissible. In this case, and due the trace theorem (A.531), if $v \in H^1(\Omega_i)$ or $v \in W^1(\Omega_e)$, then the trace of v fulfills

$$\gamma_0 v = v|_\Gamma \in H^{1/2}(\Gamma). \tag{D.168}$$

Moreover, the trace of the normal derivative can be also defined, and it holds that

$$\gamma_1 v = \frac{\partial v}{\partial n}|_\Gamma \in H^{-1/2}(\Gamma). \tag{D.169}$$

D.9.2 Regularity of the integral operators

The boundary integral operators (D.63), (D.64), (D.65), and (D.66) can be characterized as linear and continuous applications such that

$$S : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \qquad D : H^{1/2+s}(\Gamma) \longrightarrow H^{3/2+s}(\Gamma), \tag{D.170}$$

$$D^* : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \qquad N : H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma). \tag{D.171}$$

This result holds for any $s \in \mathbb{R}$ if the boundary Γ is of class C^∞, which can be derived from the theory of singular integral operators with pseudo-homogeneous kernels (cf., e.g., Nédélec 2001). Due the compact injection (A.554), it holds also that the operators

$$D : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma) \qquad \text{and} \qquad D^* : H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma) \tag{D.172}$$

are compact. For a strong Lipschitz boundary $\Gamma \in C^{0,1}$, on the other hand, these results hold only when $|s| < 1$ (cf. Costabel 1988). In the case of more regular boundaries, the range for s increases, but remains finite. For our purposes we use $s = 0$, namely

$$S : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \qquad D : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \tag{D.173}$$

$$D^* : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \qquad N : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \tag{D.174}$$

which are all linear and continuous operators, and where the operators D and D^* are compact. Similarly, we can characterize the single and double layer potentials defined respectively in (D.58) and (D.59) as linear and continuous integral operators such that

$$\mathcal{S} : H^{-1/2}(\Gamma) \longrightarrow W^1(\Omega_e \cup \Omega_i) \qquad \text{and} \qquad \mathcal{D} : H^{1/2}(\Gamma) \longrightarrow W^1(\Omega_e \cup \Omega_i). \tag{D.175}$$

D.9.3 Application to the integral equations

It is not difficult to see that if $\mu \in H^{1/2}(\Gamma)$ and $\nu \in H^{-1/2}(\Gamma)$ are given, then the transmission problem (D.35) admits a unique solution $u \in W^1(\Omega_e \cup \Omega_i)$, as a consequence of the integral representation formula (D.47). For the direct perturbation problem (D.8), though, this is not always the case, as was appreciated in the exterior sphere problem (D.140). Nonetheless, if the Fredholm alternative applies, then we know that the existence and uniqueness of the problem can be ensured almost always, i.e., except on a countable set of values for the impedance.

We consider an impedance $Z \in L^\infty(\Gamma)$ and an impedance data function $f_z \in H^{-1/2}(\Gamma)$. In both cases all the continuous functions on Γ are included.

a) First extension by zero

Let us consider the first integral equation of the extension-by-zero alternative (D.103), which is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{\mu}{2} + S(Z\mu) - D(\mu) = S(f_z) \qquad \text{in } H^{1/2}(\Gamma). \tag{D.176}$$

Due the imbedding properties of Sobolev spaces and in the same way as for the full-plane impedance Laplace problem, it holds that the left-hand side of the integral equation corresponds to an identity and two compact operators, and thus Fredholm's alternative applies.

b) Second extension by zero

The second integral equation of the extension-by-zero alternative (D.107) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{Z}{2}\mu - N(\mu) + D^*(Z\mu) = \frac{f_z}{2} + D^*(f_z) \qquad \text{in } H^{-1/2}(\Gamma). \tag{D.177}$$

The operator N plays the role of the identity and the other terms on the left-hand side are compact, thus Fredholm's alternative holds.

c) Continuous impedance

The integral equation of the continuous-impedance alternative (D.115) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$-N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu) = f_z \qquad \text{in } H^{-1/2}(\Gamma). \tag{D.178}$$

Again, the operator N plays the role of the identity and the remaining terms on the left-hand side are compact, thus Fredholm's alternative applies.

d) Continuous value

The integral equation of the continuous-value alternative (D.123) is given in terms of boundary layer potentials, for $\nu \in H^{-1/2}(\Gamma)$, by

$$\frac{\nu}{2} + ZS(\nu) - D^*(\nu) = -f_z \qquad \text{in } H^{-1/2}(\Gamma). \tag{D.179}$$

On the left-hand side we have an identity operator and the remaining operators are compact, thus Fredholm's alternative holds.

e) Continuous normal derivative

The integral equation of the continuous-normal-derivative alternative (D.131) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{Z}{2}\mu - N(\mu) + ZD(\mu) = f_z \qquad \text{in } H^{-1/2}(\Gamma). \tag{D.180}$$

As before, Fredholm's alternative again applies, since on the left-hand side we have the operator N and two compact operators.

D.9.4 Consequences of Fredholm's alternative

Since the Fredholm alternative applies to each integral equation, therefore it applies also to the exterior differential problem (D.8) due the integral representation formula. The existence of the exterior problem's solution is thus determined by its uniqueness, and the impedances $Z \in \mathbb{C}$ for which the uniqueness is lost constitute a countable set, which we call the impedance spectrum of the exterior problem and denote it by σ_Z. The existence and uniqueness of the solution is therefore ensured almost everywhere. The same holds obviously for the solution of the integral equation, whose impedance spectrum we denote by ς_Z. Since each integral equation is derived from the exterior problem, it holds that $\sigma_Z \subset \varsigma_Z$. The converse, though, is not necessarily true and depends on each particular integral equation. In any way, the set $\varsigma_Z \setminus \sigma_Z$ is at most countable.

Fredholm's alternative applies as much to the integral equation itself as to its adjoint counterpart, and equally to their homogeneous versions. Moreover, each integral equation solves at the same time an exterior and an interior differential problem. The loss of uniqueness of the integral equation's solution appears when the impedance Z is an eigenvalue of some associated interior problem, either of the homogeneous integral equation or of its adjoint counterpart. Such an impedance Z is contained in ς_Z.

The integral equation (D.105) is associated with the extension by zero (D.98), for which no eigenvalues appear. Nevertheless, its adjoint integral equation (D.125) of the continuous value is associated with the interior problem (D.118), whose solution is unique for all $Z \neq 0$.

The integral equation (D.108) is also associated with the extension by zero (D.98), for which no eigenvalues appear. Nonetheless, its adjoint integral equation (D.133) of the continuous normal derivative is associated with the interior problem (D.126), whose solution is unique for all Z, without restriction.

The integral equation (D.117) of the continuous impedance is self-adjoint and is associated with the interior problem (D.109), which has a countable quantity of eigenvalues Z.

Let us consider now the transmission problem generated by the homogeneous exterior problem

$$\begin{cases} \text{Find } u_e : \Omega_e \to \mathbb{C} \text{ such that} \\ \Delta u_e = 0 & \text{in } \Omega_e, \\ -\dfrac{\partial u_e}{\partial n} + Z u_e = 0 & \text{on } \Gamma, \\ + \text{ Decaying condition as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{D.181}$$

and the associated homogeneous interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i = 0 & \text{in } \Omega_i, \\ \dfrac{\partial u_i}{\partial n} + Z u_i = 0 & \text{on } \Gamma, \end{cases} \tag{D.182}$$

where the asymptotic decaying condition is as usual given by (D.5), and where the unit normal $\boldsymbol{n}$ always points outwards of Ω_e.

As in the two-dimensional case, it holds again that the integral equations for this transmission problem have either the same left-hand side or are mutually adjoint to all other possible alternatives of integral equations that can be built for the exterior problem (D.8), and in particular to all the alternatives that were mentioned in the last subsection. The eigenvalues Z of the homogeneous interior problem (D.182) are thus also contained in ς_Z.

We remark that additional alternatives for integral representations and equations based on non-homogeneous versions of the problem (D.182) can be also derived for the exterior impedance problem (cf. Ha-Duong 1987).

The determination of the impedance spectrum σ_Z of the exterior problem (D.8) is not so easy, but can be achieved for simple geometries where an analytic solution is known.

In conclusion, the exterior problem (D.8) admits a unique solution u if $Z \notin \sigma_Z$, and each integral equation admits a unique solution, either μ or ν, if $Z \notin \varsigma_Z$.

D.10 Variational formulation

To solve a particular integral equation we convert it to its variational or weak formulation, i.e., we solve it with respect to certain test functions in a bilinear (or sesquilinear) form. Basically, the integral equation is multiplied by the (conjugated) test function and then the equation is integrated over the boundary of the domain. The test functions are taken in the same function space as the solution of the integral equation.

a) First extension by zero

The variational formulation for the first integral equation (D.176) of the extension-by-zero alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{\mu}{2} + S(Z\mu) - D(\mu), \varphi \right\rangle = \big\langle S(f_z), \varphi \big\rangle. \tag{D.183}$$

b) Second extension by zero

The variational formulation for the second integral equation (D.177) of the extension-by-zero alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{Z}{2}\mu - N(\mu) + D^*(Z\mu), \varphi \right\rangle = \left\langle \frac{f_z}{2} + D^*(f_z), \varphi \right\rangle. \tag{D.184}$$

c) Continuous impedance

The variational formulation for the integral equation (D.178) of the alternative of the continuous-impedance searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\big\langle -N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu), \varphi \big\rangle = \big\langle f_z, \varphi \big\rangle. \tag{D.185}$$

d) Continuous value

The variational formulation for the integral equation (D.179) of the continuous-value alternative searches $\nu \in H^{-1/2}(\Gamma)$ such that $\forall \psi \in H^{-1/2}(\Gamma)$

$$\left\langle \frac{\nu}{2} + ZS(\nu) - D^*(\nu), \psi \right\rangle = \big\langle -f_z, \psi \big\rangle. \tag{D.186}$$

e) Continuous normal derivative

The variational formulation for the integral equation (D.180) of the continuous-normal-derivative alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{Z}{2}\mu - N(\mu) + ZD(\mu), \varphi \right\rangle = \big\langle f_z, \varphi \big\rangle. \tag{D.187}$$